# Equivariant K-theory of compactifications of algebraic groups

V.Uma

**Abstract**

In this article we describe the $G\times G$-equivariant $K$-ring of $X$, where $X$ is a regular compactification of a connected complex reductive algebraic group $G$. Furthermore, in the case when $G$ is a semisimple group of adjoint type, and $X$ its wonderful compactification, we describe its ordinary $K$-ring $K(X)$. More precisely, we prove that $K(X)$ is a free module over $K(G/B)$ of rank the cardinality of the Weyl group. We further give an explicit basis of $K(X)$ over $K(G/B)$, and also determine the structure constants with respect to this basis.

## Introduction

Let $G$ denote a connected complex reductive algebraic group, $B\subset G$ a Borel subgroup and $T\subset B$ a maximal torus of dimension $l$. Let $C$ be the center of $G$ and let $G_{ad}:=G/C$ be the corresponding semisimple adjoint group. Let $W$ denote the Weyl group of $(G,T)$.

A normal complete variety $X$ is called an *equivariant compactification* of $G$ if $X$ contains $G$ as an open subvariety and the action of $G\times G$ on $G$ by left and right multiplication extends to $X$. We say that $X$ is a *regular compactification* of $G$ if $X$ is an equivariant compactification of $G$ which is regular as a $G\times G$-variety ( [4, Section 2.1]). Smooth complete toric varieties are regular compactifications of the torus. For the adjoint group $G_{ad}$, the wonderful compactification $\overline{G_{ad}}$ constructed by De Concini and Procesi in [10] is the unique regular compactification of $G_{ad}$ with a unique closed $G_{ad}\times G_{ad}$-orbit.

The main aim of this article is to describe the $T\times T$-equivariant and $G\times G$-equivariant $K$-ring of $X$. For this purpose, we essentially follow the methods used in the description of the $T\times T$-equivariant and $G\times G$-equivariant Chow ring of $X$ by Brion ([4, Section 3]). Indeed, we see that these methods can be naturally generalised to the setting of $K$-theory, for the purpose of which we apply as key tools, the localisation theorem of Vezzosi and Vistoli ([23, Theorem 2]) and the results of Merkurjev ([19, Theorem 4.2]).

We begin with a Preliminary section §1, where we recall basic notions on

equivariant $K$-theory and prove certain necessary facts which are later used in proving the main results. We refer to §1 and §2 for the notations used below.

In §2 (see Theorem 2.1) we describe $K_{T\times T}(X)$ in terms of closed $G\times G$-orbits and the $T\times T$-invariant curves described in [4, Section 3]. More precisely, using the localisation theorem we embed $K_{T\times T}(X)$ inside $\prod_{\sigma\in\mathcal{F}_+(l)} K_{T\times T}(Z_\sigma)$, where each $Z_\sigma \simeq G/B^- \times G/B$ is a closed $G\times G$-orbit in $X$. The image of $K_{T\times T}(X)$ inside $\prod_{\sigma\in\mathcal{F}_+(l)} K_{T\times T}(Z_\sigma)$ is further described by certain equivalence relations which are completely determined by the $T\times T$-action on the $T\times T$-invariant curves joining the $T\times T$-fixed points, which are the base points of the closed orbits.

Using the above, we further get a description of $K_{G\times G}(X)$ in Cor. 2.2 and Cor. 2.3. In particular, we prove in Cor. 2.3 that $K_{G\times G}(X) \simeq (K_{T\times T}(X))^W \simeq (K_T(\overline{T}) \otimes R(T))^W$, where $\overline{T}$ denotes the closure of $T$ in $X$. As a consequence, $K_{G\times G}(X)$ is a module over its subring $R(T)\otimes R(T)^W \simeq R(T)\otimes R(G)$. Here we mention that Cor. 2.3 is analogous to the corresponding result for equivariant cohomology of wonderful compactifications due to Littelmann and Procesi ([16]).

In Theorem 2.10 we give an explicit description of the additive structure of $K_{G\times G}(X)$ as a module over its subring $1\otimes R(G)$. More precisely, we give a direct sum decomposition of $K_{G\times G}(X)$, where each piece of the decomposition is a $1\otimes R(G)$-submodule of the ring $K_T(\overline{T}^+)\otimes R(T)$ (see §2 for the definition of the toric variety $\overline{T}^+$). Further, by defining the multiplication of the pieces inside the subring $K_T(\overline{T}^+)\otimes R(T)$ we describe the ring structure of $K_{G\times G}(X)$, and obtain the explicit multiplication rule (see Cor. 2.12). Moreover, from the direct sum decomposition we also get a natural multifiltration (see Cor. 2.11) where the filtered pieces are $R(T)\otimes R(G)$-submodules.

The rational equivariant cohomology of regular embeddings of symmetric spaces have been described by Bifet, De Concini and Procesi ([2]) in terms of Stanley-Reisner systems. Our approach via the localisation theorem yields another proof of their results for group embeddings, and also an integral version via $K$-theory.

In §3, we take $G$ to be the simply connected cover of the semisimple adjoint group $G_{ad}$, and $T$ a maximal torus of $G$. Then *for the wonderful compactification* $X$ of $G_{ad}$, we give a direct sum decomposition of $K_{G\times G}(X)$ as a free module of rank $|W|$ over the subring $R(T)\otimes R(G)$ (see Theorem 3.3). Moreover, each piece of the direct sum is canonically isomorphic to submodules of $R(T)\otimes R(T)$. This enables us to describe the multiplication of the direct sum pieces inside the subring $R(T)\otimes R(T)$. We also give an explicit description of the multiplicative structure and the multiplication rule of the basis elements (see Theorem 3.8).

Finally, by further application of the result of Merkurjev ([19, Theorem 4.2]), we describe the ordinary $K$-ring of $X$. More precisely, we prove that the subring generated by $Pic(X)$ in $K(X)$ is canonically isomorphic to $K(G/B)$, and $K(X)$ is a free module of rank $|W|$ over this subring. Furthermore, we also give a

precise description of the multiplication of the basis elements over $K(G/B)$ in ordinary $K$-ring by pushing down the multiplicative structure in the equivariant $K$-ring. More precisely, in Theorem 3.12 we construct an explicit basis of $K(X)$ over $K(G/B)$ and determine the structure constants with respect to this basis.

**Acknowledgements:** This work was carried out during my one year stay at Institut Fourier, Grenoble under the French Government post-doctoral fellowship. I am grateful to Michel Brion for his insightful guidance and for the invaluable discussions we have had throughout the period of this work at Institut Fourier. I also thank him for a careful reading of earlier versions of this manuscript and giving several suggestions for improving its presentation. I thank V. Balaji for valuable discussions and for constant encouragement during this work. I thank both the referees for their careful reading of the manuscript and for their comments and valuable suggestions which has led to improving the exposition.

# 1 Preliminaries

## 1.1 Regular group compactifications

Let $W$ denote the Weyl group and $\Phi$ denote the root system of $(G,T)$. We have the subset $\Phi^+$ of positive roots and its subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots where $r$ is the semisimple rank of $G$. For $\alpha \in \Delta$ we denote by $s_\alpha$ the corresponding simple reflection. For any subset $I \subset \Delta$, let $W_I$ denote the subgroup of $W$ generated by all $s_\alpha$ for $\alpha \in I$. At the extremes we have $W_\emptyset = \{1\}$ and $W_\Delta = W$.

A $G$-*variety* is a complex algebraic variety with an algebraic action of $G$.

We now recall the definition of a regular $G$-variety due to Bifet, De Concini and Procesi. (see §3 of [2] and §1.4 of [4]).

**Definition 1.1.** *A $G$-variety $X$ is said to be regular if it satisfies the following conditions:*

*(i) $X$ is smooth and contains a dense $G$-orbit $X_G^0$ whose complement is a union of irreducible smooth divisors with normal crossings (the boundary divisors).*

*(ii) Any $G$-orbit closure in $X$ is the transversal intersection of the boundary divisors which contain it.*

*(iii) For any $x \in X$, the normal space $T_xX/T_x(Gx)$ contains a dense orbit of the isotropy group $G_x$.*

Consider the connected reductive group $G$ as a homogeneous space under $G \times G$ for the action given by left and right multiplication: $(g_1, g_2)\gamma = g_1 \gamma g_2^{-1}$. Then the isotropy group of the identity is the diagonal $diag(G)$.

A normal complete variety $X$ is called an equivariant *compactification* of $G$, if $X$ contains $G$ as an open subvariety and the action of $G \times G$ on $G$ by left and right multiplication extends to $X$.

We now recall the definition of a regular compactification of $G$ (see §2.1 of [4]).

**Definition 1.2.** *We say that $X$ is a regular compactification of $G$ if $X$ is a $G \times G$-equivariant compactification of $G$ which is regular as a $G \times G$-variety.*

**Examples:**

1. Smooth complete toric varieties are regular compactifications of the torus.
2. For the adjoint group $G_{ad}$, the wonderful compactification $\overline{G}_{ad}$ constructed by De Concini and Procesi in [10] is the unique regular compactification of $G_{ad}$ with a unique closed $G_{ad} \times G_{ad}$-orbit.

## 1.2 Preliminaries on K-theory

Let $X$ be a smooth projective complex $G$-variety. Let $K_G(X)$ and $K_T(X)$ denote the Grothendieck groups of $G$ and $T$-equivariant coherent sheaves on $X$ respectively. Recall that $K_T(pt) = R(T)$ and $K_G(pt) = R(G)$ where $R(T)$ and $R(G)$ denote respectively the Grothendieck group of complex representations of $T$ and $G$. The Grothendieck group of equivariant coherent sheaves can be identified with the Grothendieck ring of equivariant vector bundles on $X$. Further, the structure morphism $X \to Spec\, \mathbb{C}$ induces canonical $R(G)$ and $R(T)$-module structures on $K_G(X)$ and $K_T(X)$ respectively (see Prop. 5.1.28 of [8]) and Example 2.1 of [19]).

Let $\Lambda := X^*(T)$. Then $R(T)$ (the representation ring of the torus $T$) is isomorphic to the group algebra $\mathbb{Z}[\Lambda]$. Let $e^\lambda$ denote the element of $\mathbb{Z}[\Lambda] = R(T)$ corresponding to a weight $\lambda \in \Lambda$. Then $(e^\lambda)_{\lambda \in \Lambda}$ is a basis of the $\mathbb{Z}$ module $\mathbb{Z}[\Lambda]$. Further, since $W$ acts on $X^*(T)$, on $\mathbb{Z}[\Lambda]$ we have the following natural action of $W$ given by : $w(e^\lambda) = e^{w(\lambda)}$ for each $w \in W$ and $\lambda \in \Lambda$. Recall that we can identify $R(G)$ with $R(T)^W$ via restriction to $T$, where $R(T)^W$ denotes the subring of $R(T)$ invariant under the action of $W$.

The following is a theorem analogous to Theorem 3.4 of [5], which we shall use to prove the main result.

**Theorem 1.3.** *Let $X$ be a nonsingular projective variety on which $T$ acts with finitely many fixed points $x_1, \ldots, x_m$ and finitely many invariant curves. Then the image of*

$$\iota^* : K_T(X) \to K_T(X^T)$$

*is the set of all* $(f_1,\ldots,f_m)\in R(T)^m$ *such that* $f_i\equiv f_j \pmod{(1-e^{-\chi})}$ *whenever* $x_i$ *and* $x_j$ *lie in an invariant irreducible curve* $C$ *and* $T$*-acts on* $C$ *through the character* $\chi$.

*Proof:* By Theorem 2 of [23] it follows that the above restriction homomorphism $\iota^*$ is injective and its image is equal to the intersection of all the images of the restriction homomorphisms $K_T(X^{T'})\to K_T(X^T)$ for all subtori $T'\subseteq T$ of codimension 1.

Since $X$ contains finitely many invariant curves, $X^{T'}$ is at most one dimensional for every codimension 1 subtorus $T'\subset T$.

Let $X_{n-1}:=\bigcup X^{T'}$, where the union runs over all subtori $T'$ of codimension one in $T$. Since $X_{n-1}$ is one-dimensional it consists of disjoint union of points and nonsingular irreducible curves; let $C$ be such a curve.

If $C$ contains a unique fixed point $x$, then $\iota_x^*: K_T(C)\to K_T(x)=R(T)$ is an isomorphism. Otherwise, $C$ is isomorphic to $\mathbb{P}^1$. It follows that $C$ contains two distinct fixed points $x$ and $y$. Moreover, the image of

$$\iota_C^*: K_T(C)\to K_T(x)\times K_T(y)=R(T)\times R(T)$$

consists of pairs of elements $(f,g)\in R(T)\times R(T)$ such that $f\equiv g \pmod{(1-e^{-\chi})}$ where $T$ acts on $C$ through the weight $\chi$. This can be seen as follows:

Let us choose as a basis of $K_T(C)$ over $R(T)$ the class of the trivial line bundle $\mathcal{O}_C$, and the class of the Hopf bundle $H$ which is the dual of the tautological bundle. Then under $\iota_C^*$, the image of $\mathcal{O}_C$ is $(1,1)$ and that of $H$ is $(e^{\chi},1)$. Since any element in $K_T(C)$ is a linear combination of $\mathcal{O}_C$ and $H$, the difference of the coordinates of the image in $K_T(C^T)$ is always divisible by $1-e^{-\chi}$. Conversely, let $(f,g)\in R(T)\times R(T)$ be such that $(1-e^{-\chi})\cdot h=f-g$ for some $h\in R(T)$. Then we see that the element $g[\mathcal{O}_C]+e^{-\chi}\cdot h([H]-[\mathcal{O}_C])$ in $K_T(C)$ maps to $(f,g)$ under $\iota_C^*$.

The theorem now follows by applying Theorem 2 of [23]. □

Recall from Cor. 3.7 of [14] that there exists an exact sequence:

$$1\to\mathcal{Z}\to\widetilde{G}:=\widetilde{C}\times G^{ss}\xrightarrow{\pi}G\to 1 \tag{1.1}$$

where $\mathcal{Z}$ is a finite central subgroup, $\widetilde{C}$ is a torus and $G^{ss}$ is semisimple and simply-connected. The condition that $G^{ss}$ is simply connected implies that $\widetilde{G}$ is *factorial* (see [19]).

Then $\widetilde{B}:=\pi^{-1}(B)$ and $\widetilde{T}:=\pi^{-1}(T)$ are respectively a Borel subgroup and a maximal torus of $\widetilde{G}$. Further, by restricting the map $\pi$ to $\widetilde{T}$ we get the following exact sequence:

$$1\to\mathcal{Z}\to\widetilde{T}\to T\to 1. \tag{1.2}$$

Let $\widetilde{W}$ and $\widetilde{\Phi}$ denote respectively the Weyl group and the root system of $(\widetilde{G},\widetilde{T})$. Then by the exact sequence (1.1) it follows in particular that $\widetilde{W}=W$ and $\widetilde{\Phi}=\Phi$.

Further we have $R(\widetilde{G})=R(\widetilde{C})\otimes R(G^{ss})$ and $R(\widetilde{T})\simeq R(\widetilde{C})\otimes R(T^{ss})$ where $T^{ss}$ is the (unique) maximal torus $\widetilde{T}\cap G^{ss}$.

Recall we can identify $R(\widetilde{G})$ with $R(\widetilde{T})^W$ via restriction to $\widetilde{T}$, and further $R(\widetilde{T})$ is a free $R(\widetilde{G})$ module of rank $|W|$ (see Theorem 6.41, pp.164 of [1] and Theorem 1, pp. 199 of [20]). Moreover, since $G^{ss}$ is semi-simple and simply connected, $R(G^{ss})\simeq\mathbb{Z}[x_1,\ldots,x_r]$ is a polynomial ring on the fundamental representations. Hence $R(\widetilde{G})=R(\widetilde{C})\otimes R(G^{ss})$ is the tensor product of a polynomial ring and a Laurent polynomial ring, and hence a regular ring of dimension $r+dim(\widetilde{C})=rank(G)$ where $r$ is the rank of $G^{ss}$.

We shall consider the $\widetilde{T}$ and $\widetilde{G}$-equivariant $K$-theory of $X$ where we take the natural actions of $\widetilde{T}$ and $\widetilde{G}$ on $X$ through the canonical surjections to $T$ and $G$ respectively.

We consider $\mathbb{Z}$ as an $R(\widetilde{G})$-module by the augmentation map $\epsilon: R(\widetilde{G})\to\mathbb{Z}$ which maps any $\widetilde{G}$-representation $V$ to $dim(V)$. Moreover, we have the natural restriction homomorphisms $K_{\widetilde{G}}(X)\to K_{\widetilde{T}}(X)$ and $K_{\widetilde{G}}(X)\to K(X)$ where $K(X)$ denotes the ordinary Grothendieck ring of algebraic vector bundles on $X$. We then have the following isomorphisms (see Prop. 4.1 and Theorem 4.2 of [19]) (also see Theorem 6.1.22 pp.310 of [8]):

$(a)$ $R(\widetilde{T})\otimes_{R(\widetilde{G})}K_{\widetilde{G}}(X)\simeq K_{\widetilde{T}}(X)$.

$(b)$ $K_{\widetilde{G}}(X)\simeq K_{\widetilde{T}}(X)^W$.

$(c)$ $\mathbb{Z}\otimes_{R(\widetilde{G})}K_{\widetilde{G}}(X)\simeq K(X)$.

**Remark 1.4.** In fact the above isomorphisms $(a)$ and $(b)$ hold in higher equivariant K-theory and the isomorphism $(c)$ corresponds to the degeneration of the Merkurjev spectral sequence $E^2_{p,q}=Tor_p^{R(\widetilde{G})}(\mathbb{Z},K^q_{\widetilde{G}}(X))\Rightarrow K_{p+q}(X)$ (see pp. 2-3 of [19]).

**Remark 1.5.** We will prove in Theorem 1.8 that the isomorphism $(b)$ also holds when $\widetilde{G}$ and $\widetilde{T}$ are replaced with $G$ and $T$ respectively i.e, $K_G(X)\simeq K_T(X)^W$.

**Lemma 1.6.** *Let $X$ be a smooth projective $G$-variety containing only finitely many $T$-fixed points. Then $K_T(X)$ is a free module over $R(T)$ of rank $|X^T|$. Furthermore, $K_{\widetilde{T}}(X)$ (resp. $K_{\widetilde{G}}(X)$) is also free over $R(\widetilde{T})$ (resp. $R(\widetilde{G})$) of rank $|X^T|$.*

*Proof:* Since $X$ is a smooth projective variety with $T$-action such that $X^T$ is finite, it admits a Bialynicki-Birula cellular decomposition with $m = |X^T|$ $T$-stable affine cells. Let $X_1, X_2, \ldots, X_m = pt$ be an ordering of the cells with $dim(X_1) \geq dim(X_2) \geq \ldots$. Set $X^j = \sqcup_{i \geq j} X_i$. Then $X = X^0 \supset X^1 \supset \cdots \supset X^m = pt$ is a decreasing filtration on $X$ by closed $T$-stable subvarieties. Thus $X \to pt$ is a $T$-equivariant cellular fibration over a point. Therefore by the Cellular Fibration Lemma (see pp. 270 [8]) it follows that, $K_T(X)$ is free module over $K_T(pt) = R(T)$ of rank $m$.

Since $\widetilde{T}$ acts on $X$ via the canonical surjection to $T$ it similarly follows that $K_{\widetilde{T}}(X)$ is free module over $K_{\widetilde{T}}(pt) = R(\widetilde{T})$ of rank $|X^{\widetilde{T}}| = |X^T| = m$.

Now, since $K_{\widetilde{T}}(X)$ is a free module over $R(\widetilde{T})$, and $R(\widetilde{T})$ is free over $R(\widetilde{G})$, it follows that $K_{\widetilde{T}}(X)$ is a free module over $R(\widetilde{G})$. Further, since $R(\widetilde{G})$ is a direct summand of $R(\widetilde{T})$, the isomorphism $(a)$ above implies that $K_{\widetilde{G}}(X)$ is a direct summand of $K_{\widetilde{T}}(X)$ as an $R(\widetilde{G})$-module. Thus $K_{\widetilde{G}}(X)$ is a projective module over $R(\widetilde{G})$. Moreover, since $R(\widetilde{G})$ is a tensor product of a polynomial ring and a Laurent polynomial ring $K_{\widetilde{G}}(X)$ is in fact free over $R(\widetilde{G})$ (see Theorem 1.1 of [12]).

The isomorphism $(a)$ above further implies that the rank of $K_{\widetilde{G}}(X)$ over $R(\widetilde{G})$ is same as the rank of $K_{\widetilde{T}}(X)$ over $R(\widetilde{T})$ which is $m$. $\square$

**Lemma 1.7.** *Let $1 \to \mathcal{Z} \to \widetilde{T} \to T \to 1$ be an exact sequence of algebraic groups where $\widetilde{T}$ and $T$ are complex tori and $\mathcal{Z}$ is a finite abelian group. Let $1_{\mathcal{Z}} = \chi_1, \ldots, \chi_m$ denote the characters of $\mathcal{Z}$. Further, let $1_{\widetilde{T}} = \widetilde{\chi_1}, \ldots, \widetilde{\chi_m}$ be arbitrary lifts of $\chi_1, \ldots, \chi_m$ to characters of $\widetilde{T}$. Let $Y$ be an irreducible $T$-variety. We then have the following isomorphisms:*

*(i)* $R(\widetilde{T}) \simeq \bigoplus_{i=1}^m R(T) e^{\widetilde{\chi_i}}$

*(ii)* $K_{\widetilde{T}}(Y) \simeq K_T(Y) \otimes_{R(T)} R(\widetilde{T})$

*Proof:* Let $V$ be any $\widetilde{T}$-representation and $V = \bigoplus_{\chi \in X^*(\widetilde{T})} V_\chi$ be the direct sum decomposition of $V$ as $\widetilde{T}$-weight spaces. Then

$$V_i := \bigoplus_{\chi|_{\mathcal{Z}} = \chi_i} V_\chi.$$

are the isotypical components with respect to the characters $\chi_1, \ldots, \chi_m$ of $\mathcal{Z}$. Thus as $\widetilde{T}$-modules we have an isomorphism $V_i \simeq e^{\widetilde{\chi_i}} \otimes V^i$, where the $\widetilde{T}$-module $V^i$ is in fact a $T$-module since the $\mathcal{Z}$-action on it is trivial. Since $V = \bigoplus_{i=1}^m V_i$, it follows that as $\widetilde{T}$-modules we have $V \simeq \bigoplus_{i=1}^m e^{\widetilde{\chi_i}} \otimes V^i$. This proves $(i)$.

We have a canonical homomorphism of rings $R(\widetilde{T}) \otimes_{R(T)} K_T(Y) \to K_{\widetilde{T}}(Y)$ where, $[V] \in R(\widetilde{T})$ maps to the trivial bundle $Y \times V$ and the map from $K_T(Y)$

to $K_{\widetilde{T}}(Y)$ is induced by the surjection $\widetilde{T} \to T$. To define the inverse of the above homomorphism:

Let $E$ be a $\widetilde{T}$-equivariant vector bundle on $Y$. Since $Y$ is a $T$-variety, the $\mathcal{Z}$-action on $Y$ is trivial. Thus on every fibre of $E$ we get a canonical linear $\mathcal{Z}$-action, which gives a weight space decomposition on each fiber. Note that since $\mathcal{Z}$ is finite, the weights of $\mathcal{Z}$ form a finite set. Moreover, since $E$ is locally trivial the $\mathcal{Z}$-representation is locally constant and hence globally constant over the irreducible base $Y$. Thus we get the following vector bundle direct sum decomposition

$$E = \bigoplus_{i=1}^{m} E_i$$

where $E_i$ denotes the subbundle whose fibre is the eigenspace corresponding to the character $\chi_i$ of $\mathcal{Z}$. Thus as $\widetilde{T}$-equivariant bundles we have an isomorphism $E_i \simeq e^{\widetilde{\chi_i}} \otimes E^i$, where the $\widetilde{T}$-equivariant bundle $E^i$ is in fact a $T$-equivariant bundle since the $\mathcal{Z}$-action on it is trivial.

Therefore the inverse map is defined by sending $E$ to the element $\bigoplus_{i=1}^{m} e^{\widetilde{\chi_i}} \otimes E^i$ of $R(\widetilde{T}) \otimes_{R(T)} K_T(Y)$. This proves $(ii)$.

**Theorem 1.8.** *The restriction homomorphism $K_G(X) \to K_T(X)$ induces an isomorphism $K_G(X) \simeq K_T(X)^W$ where $K_T(X)^W$ denotes the subring of $W$-invariants of $K_T(X)$. (For the corresponding result in topological $K$-theory see [18, Theorem 4.4]).*

*Proof:* Recall (see Prop. 2.10 of [19]) that we have the following isomorphism:

$$K_T(X) \simeq K_G(X \times G/B).$$

Therefore the projection $p : X \times G/B \to X$ induces the pull-back map $p^* : K_G(X) \to K_T(X)$. Note that $p$ is a proper map since its fibre $G/B$ is complete. Thus we further have the push-forward map $p_* : K_T(X) \to K_G(X)$. Now by the projection formula we get $p_* \circ p^* = id$ (see 5.2.13 and 5.3.12 of [8]). In particular, it follows that $p^*$ is injective.

Similarly, the projection $\widetilde{p} : X \times \widetilde{G}/\widetilde{B} \to X$ induces the pull-back and push-forward maps: $\widetilde{p}^* : K_{\widetilde{G}}(X) \to K_{\widetilde{T}}(X)$ and $\widetilde{p}_* : K_{\widetilde{T}}(X) \to K_{\widetilde{G}}(X)$ respectively. Further, by the projection formula we get $\widetilde{p}_* \circ \widetilde{p}^* = id$, and hence $\widetilde{p}^*$ is injective. Furthermore, by the isomorphism $(b)$ above we know that the image of $K_{\widetilde{G}}(X)$ under $\widetilde{p}^*$ is $K_{\widetilde{T}}(X)^W$.

Let $u := \pi : \widetilde{G} \to G$ and $v := \pi |_{\widetilde{T}}: \widetilde{T} \to T$. Then $u$ and $v$ induce the ring homomorphisms: $u^* : K_G(X) \to K_{\widetilde{G}}(X)$ and $v^* : K_T(X) \to K_{\widetilde{T}}(X)$ respectively. Further, since the isomorphism $K_T(X) \simeq K_G(X \times G/B)$ is canonical, $\pi^* : K_G(X \times G/B) \to K_{\widetilde{G}}(X \times \widetilde{G}/\widetilde{B})$ can be identified with $v^*$.

Now, by (ii) of Lemma 1.7 above, the map $v^* : K_T(X) \hookrightarrow K_{\widetilde{T}}(X)$ induced by the surjection $\widetilde{T} \to T$ is injective. We now claim that $v^*p^* = \widetilde{p}^*u^*$, so that $u^*$ is also injective. This can be seen as follows:

For any $G$-vector bundle $V$ on $X$, let $\widetilde{V}$ denote $V$ thought of as a $\widetilde{G}$-vector bundle via the surjection $\pi : \widetilde{G} \to G$. Then we see that $p^*([V]) = [V] \boxtimes [\mathcal{O}_{G/B}] \in K_G(X \times G/B)$. Further, $u^*p^*([V]) = [\widetilde{V}] \boxtimes [\mathcal{O}_{\widetilde{G}/\widetilde{B}}] \in K_{\widetilde{G}}(X \times \widetilde{G}/\widetilde{B})$. Moreover, since $u^*([V]) = [\widetilde{V}]$, we see that $\widetilde{p}^*u^*([V]) = [\widetilde{V}] \boxtimes [\mathcal{O}_{\widetilde{G}/\widetilde{B}}] \in K_{\widetilde{G}}(X \times \widetilde{G}/\widetilde{B})$. Hence the claim.

Using the isomorphism $G/B \simeq \widetilde{G}/\widetilde{B}$, and the fact that the push forward map is functorial it follows that: $\widetilde{p}_* : K_{\widetilde{G}}(X \times \widetilde{G}/\widetilde{B}) \to K_{\widetilde{G}}(X)$ restricts to $p_* : K_G(X \times G/B) \to K_G(X)$. That is, $\widetilde{p}_*v^* = u^*p_*$.

Thus we get the following commuting diagram:

$$\begin{array}{ccc} K_G(X) & \overset{u^*}{\hookrightarrow} & K_{\widetilde{G}}(X) \\ {\scriptstyle p_*}\uparrow\downarrow{\scriptstyle p^*} & & {\scriptstyle \widetilde{p}_*}\uparrow\downarrow{\scriptstyle \widetilde{p}^*} \\ K_T(X) & \overset{v^*}{\hookrightarrow} & K_{\widetilde{T}}(X) \end{array}$$

In particular, it follows from the above diagram that $p^*(K_G(X)) \subseteq K_T(X)^W$. Hence it remains to show that $K_T(X)^W \subseteq p^*(K_G(X))$. This can be seen as follows:

Let $\alpha \in K_T(X)^W$. Then $v^*\alpha \in K_{\widetilde{T}}(X)^W$. Further, let $v^*\alpha = \widetilde{p}^*\beta$ for $\beta \in K_{\widetilde{G}}(X)$. Thus $u^*p_*\alpha = \widetilde{p}_*v^*\alpha = \widetilde{p}_*\widetilde{p}^*\beta = \beta$.

Now to show that $\alpha = p^*(\gamma)$ for $\gamma \in K_G(X)$. If this is true, this further implies that $\gamma = p_*(\alpha)$. Therefore it is enough to show that $\alpha = p^*p_*\alpha$. Since $v^*$ is injective this is further equivalent to showing that $v^*\alpha = v^*p^*p_*\alpha$. But this follows from the above arguments, since $v^*\alpha = \widetilde{p}^*\beta = \widetilde{p}^*u^*p_*\alpha = v^*p^*p_*\alpha$. Hence the theorem. □

Let $R(\widetilde{T})^{W_I}$ denote the invariant subring of the ring $R(\widetilde{T})$ under the action of the subgroup $W_I$ of $W$ for every $I \subset \Delta$. Thus in particular we have, $R(\widetilde{T})^W = R(\widetilde{G})$ and $R(\widetilde{T})^{\{1\}} = R(\widetilde{T})$. Further, for every $I \subset \Delta$, $R(\widetilde{T})^{W_I}$ is a free module over $R(\widetilde{G}) = R(\widetilde{T})^W$ of rank $|W/W_I|$ (see Theorem 2.2 of [22]). Indeed, Theorem 2.2 of [22] which we apply here holds for $R(T^{ss})$. However, since $W$ acts trivially on the central torus $\widetilde{C}$ and hence trivially on $R(\widetilde{C})$ we have $R(\widetilde{T})^{W_I} = R(\widetilde{C}) \otimes R(T^{ss})^{W_I}$ for every $I \subseteq \Delta$, and hence we obtain the analogous statement for $R(\widetilde{T})$.

Let $W^I$ denote the set of minimal length coset representatives of the parabolic subgroup $W_I$ for every $I \subset \Delta$. Then

$$W^I := \{w \in W \mid l(wv) = l(w) + l(v) \ \forall\ v \in W_I\} = \{w \in W \mid w(\Phi_I^+) \subset \Phi^+\}$$

where $\Phi_I$ is the root system associated to $W_I$, with $I$ as the set of simple roots. Recall (see pp. 19 of [13]) that we also have:

$$W^I = \{w \in W \mid l(ws) > l(w) \ for \ all \ s \in I\}.$$

Note that $J \subseteq I$ implies that $W^{\Delta\setminus J} \subseteq W^{\Delta\setminus I}$. Let

$$C^I := W^{\Delta\setminus I} \setminus (\bigcup_{J \subsetneq I} W^{\Delta\setminus J}). \tag{1.3}$$

Let $\alpha_1, \ldots, \alpha_r$ be an ordering of the set $\Delta$ of simple roots and $\omega_1, \ldots, \omega_r$ denote the corresponding fundamental weights for the root system of $(G^{ss}, T^{ss})$. Since $G^{ss}$ is simply connected, the fundamental weights form a basis for $X^*(T^{ss})$ and hence for every $\lambda \in X^*(T^{ss})$, $e^\lambda \in R(T^{ss})$ is a monomial in the elements $e^{\omega_i} : 1 \leq i \leq r$.

In Theorem 2.2 of [22] Steinberg has defined a basis $\{f_v^I : v \in W^I\}$ of $R(T^{ss})^{W_I}$ as an $R(T^{ss})^W$-module. We recall here this definition: For $v \in W^I$ let

$$p_v := \prod_{v^{-1}\alpha_i < 0} e^{\omega_i} \in R(\widetilde{T}). \tag{1.4}$$

Then

$$f_v^I := \sum_{x \in W_I(v)\backslash W_I} x^{-1} v^{-1} p_v \tag{1.5}$$

where $W_I(v)$ denotes the stabilizer of $v^{-1}p_v$ in $W_I$.

We shall also denote by $\{f_v^I : v \in W^I\}$ the corresponding basis of $R(\widetilde{T})^{W_I}$ as an $R(\widetilde{T})^W$-module where it is understood that $f_v^I := 1 \otimes f_v^I \in R(\widetilde{C}) \otimes R(T^{ss})^{W_I}$.

We now fix the following notations before we state the next proposition.

(a) For $v \in W^I$, we shall denote by $W_I^\ell(v)$ the *minimal length representatives* of the cosets in $W_I(v)\backslash W_I$. Note that each $w \in W_I$ can be uniquely expressed as $w = ux$ where $x \in W_I^\ell(v)$ ($x$ is the unique element of smallest length in the coset $W_I(v)w$) and $u \in W_I(v)$, such that $l(w) = l(u) + l(x)$.

(b) Let $I \supseteq J$. For each $x \in W_{\Delta\setminus J}^\ell(v) \subseteq W_{\Delta\setminus J}$ we can now consider the *minimal length representative* of the coset $xW_{\Delta\setminus I} \in W_{\Delta\setminus J}/W_{\Delta\setminus I}$ which we shall denote by $x'$. Let

$$\left[W_{\Delta\setminus J}^\ell(v)\right]^{\Delta\setminus I} := \{x' \ : \ x \in W_{\Delta\setminus J}^\ell(v)\}.$$

**Proposition 1.9.** *With the above notations we have the following :*

1. *For $v \in W^{\Delta \setminus I}$ we have:*

$$f_v^{\Delta \setminus I} = \sum_{x \in W^{\ell}_{\Delta \setminus I}(v)} x^{-1} v^{-1} p_v = \sum_{x \in W^{\ell}_{\Delta \setminus I}(v)} f^{\emptyset}_{vx}$$

*where $f^{\emptyset}_{vx}$ is well defined since $vx \in W^{\emptyset} = W$.*

2. *For $v \in W^{\Delta \setminus J}$ and for $I \supseteq J$ we have:*

$$f_v^{\Delta \setminus J} = \sum_{x' \in \left[W^{\ell}_{\Delta \setminus J}(v)\right]^{\Delta \setminus I}} f^{\Delta \setminus I}_{vx'}$$

*where $f^{\Delta \setminus I}_{vx'}$ is well defined since $vx' \in W^{\Delta \setminus I}$.*

3. *For $v \in W^{\Delta \setminus I}$, $f_v^{\Delta \setminus I}$ is in the $R(\widetilde{T})^W$-span of $\{f^{\Delta \setminus J}_{v'} : v' \in C^J\}_{J \subseteq I}$.*

*Proof:* This proposition may be well known to experts but since we could not find a proof in the literature we give it below.

Let $v \in W^{\Delta \setminus I}$ and $x \in W^{\ell}_{\Delta \setminus I}(v)$. Then we claim that :

$$x^{-1} v^{-1} p_v = x^{-1} v^{-1} p_{vx}. \tag{1.6}$$

We shall prove (1.6) by induction on the length of $x$. Let $l(x) = 1$. Then $x = s_\beta$ for $\beta \in \Delta \setminus I$ such that $s_\beta \notin W_{\Delta \setminus I}(v)$. Thus we require to show that

$$s_\beta v^{-1} p_v = (v s_\beta)^{-1} p_{v s_\beta}. \tag{1.7}$$

This is equivalent to showing that $p_v = p_{v s_\beta}$ which can be seen as follows:

For $w \in W$ let $R(w) := \{\alpha \in \Phi^+ : w(\alpha) < 0\}$. Then by $(*)$ in pp. 407 of [15] it follows that:

$$R(s_\beta \cdot v^{-1}) = R(v^{-1}) \bigsqcup v R(s_\beta). \tag{1.8}$$

Note that $R(s_\beta) = \beta$. Moreover, observe that $v(\beta)$ is not a simple root (for this see below [1]). Hence by (1.8) it follows that the simple roots in $R(s_\beta v^{-1}) =$ the simple roots in $R(v^{-1})$. This by (1.4) further implies that $p_v = p_{v s_\beta}$ which proves (1.7).

Now we assume by induction that (1.6) holds for all $y \in W^{\ell}_{\Delta \setminus I}(v)$ with $l(y) < l(x)$.

[1] If $v(\beta)$ were a simple root then by (1.8) we have $v(\beta) \notin R(v^{-1})$. Thus by (1.4) we have $s_{v(\beta)} p_v = p_v$ and hence $v s_\beta v^{-1} p_v = p_v$. This implies that $s_\beta v^{-1} p_v = v^{-1} p_v$ which is a contradiction to our assumption that $s_\beta \notin W_{\Delta \setminus I}(v)$.

Let $x = ys_\beta$ be a reduced expression for $x$ where $\beta$ *is a simple root*, $y \in W_{\Delta\setminus I}$ and $l(y) = l(x) - 1$. Further, since $x \in W^{\ell}_{\Delta\setminus I}(v)$ we must have $y \notin W_{\Delta\setminus I}(v)$. Indeed it can be seen that $y \in W^{\ell}_{\Delta\setminus I}(v)$ (for this see below [2]).

Hence by induction assumption

$$y^{-1}v^{-1}p_v = y^{-1}v^{-1}p_{vy}.$$

Let $\Delta \setminus I_1 := \{\alpha \in \Delta \setminus I : l(ys_\alpha) > l(y)\}$. Then we *claim* that:

(i) $vy \in W^{\Delta\setminus I_1}$

(ii) $s_\beta \in W^{\ell}_{\Delta\setminus I_1}(vy)$

Once we prove the above *claim* we see that the equality (1.7) for $v$ replaced by $vy$ and $I$ replaced by $I_1$ will imply the equality (1.6). Thus it only remains to prove (i) and (ii) above.

*Proof of* (i): Since $v \in W^{\Delta\setminus I}$ we have: $l(vys_\alpha) = l(v)+l(ys_\alpha) > l(v)+l(y) = l(vy)$ for every $\alpha \in \Delta \setminus I_1$. Hence (i) follows.

*Proof of* (ii): Since $x = ys_\beta$ is a reduced expression, clearly $s_\beta \in W_{\Delta\setminus I_1}$. Suppose that $s_\beta \in W_{\Delta\setminus I_1}(vy)$. Since by induction we have $p_v = p_{vy}$, it follows that: $ys_\beta y^{-1} \in W_{\Delta\setminus I}(v)$. This further implies that $x = ys_\beta = zy$ for an element $z \in W_{\Delta\setminus I}(v)$. Since $l(y) \lneq l(x)$ this clearly contradicts that $x \in W^{\ell}_{\Delta\setminus I}(v)$. Thus we conclude that $s_\beta \notin W_{\Delta\setminus I_1}(vy)$ which implies (ii).

Observe that, on the right hand side of (1.5), without loss of generality we can assume that $x \in W^{\ell}_{\Delta\setminus I}(v)$. Now (1.5) and (1.6) together imply:

$$f_v^{\Delta\setminus I} = \sum_{x\in W^{\ell}_{\Delta\setminus I}(v)} x^{-1}v^{-1}p_v = \sum_{x\in W^{\ell}_{\Delta\setminus I}(v)} (vx)^{-1}p_{vx} = \sum_{W^{\ell}_{\Delta\setminus I}(v)} f^{\emptyset}_{vx} \tag{1.9}$$

which proves (1) of Prop. 1.9.

Now let $J \subseteq I$. Further, let $v \in W^{\Delta\setminus J}$ and $x \in W^{\ell}_{\Delta\setminus J}(v)$. Then we can uniquely express $x = x'y$ where $y \in W_{\Delta\setminus I}$ and $x' \in \left[W^{\ell}_{\Delta\setminus J}(v)\right]^{\Delta\setminus I}$.

Since there is no reduced expression of $x'$ ending in $s_\alpha$ for $\alpha \in \Delta \setminus I$, it follows that $vx' \in W^{\Delta\setminus I}$ (for this see below [3]).

---

[2] Suppose $y \notin W^{\ell}_{\Delta\setminus I}(v)$ then we can uniquely express $y = zy_o$ for $y_o \in W^{\ell}_{\Delta\setminus I}(v)$ and $z \in W_{\Delta\setminus I}(v)$ such that $l(y) = l(z) + l(y_o)$. Let $x_o = y_o s_\beta$ so that $x = zx_o$. Now if $l(y_o) \lneq l(y)$ then $l(x_o) \leq l(y_o) + 1 \lneq l(y) + 1 = l(x)$. This contradicts that $x \in W^{\ell}_{\Delta\setminus I}(v)$. Thus it follows that $l(y) = l(y_o)$ and hence $y = y_o$.

[3] since $l(vx's_\alpha) = l(v) + l(x's_\alpha) > l(v) + l(x') = l(vx')$

Suppose $x = x'y$ and $x_1 = x'y_1$ for $x, x_1 \in W^{\ell}_{\Delta\setminus J}(v)$ and $y, y_1 \in W_{\Delta\setminus I}$. Then we see that $x \neq x_1 \Leftrightarrow W_{\Delta\setminus I}(vx')y \neq W_{\Delta\setminus I}(vx')y_1$ (for this see below [4]).

We can now express (1.9) for $J$ as follows:

$$f_v^{\Delta\setminus J} = \sum_{x' \in \left[W^{\ell}_{\Delta\setminus J}(v)\right]^{\Delta\setminus I}} \sum_y y^{-1}(vx')^{-1} p_{vx'y} = \sum_{x' \in \left[W^{\ell}_{\Delta\setminus J}(v)\right]^{\Delta\setminus I}} f_{vx'}^{\Delta\setminus I} \tag{1.10}$$

which proves (2) of Prop. 1.9. (In the above summation $y \in W_{\Delta\setminus I}(vx')\backslash W_{\Delta\setminus I}$.)

Let $v \in W^{\Delta\setminus I}$. Then $v \in C^J$ for a unique $J \subseteq I$. Now, (1.10) can be expressed as:

$$f_v^{\Delta\setminus J} = f_v^{\Delta\setminus I} + \sum_{x' \neq 1} f_{vx'}^{\Delta\setminus I} \tag{1.11}$$

Since $v \in W^{\Delta\setminus J}$ and $x' \in W_{\Delta\setminus J}$ we have $l(vx') > l(v)$.

We now *claim* that if $v_1$ is of maximal length in $W^{\Delta\setminus I}$ then in fact $v_1 \in C^I$. This can be seen as follows:

Suppose $v_1 \notin C^I$. Let $v_1 \in W^{\Delta\setminus J}$ for $J \subsetneq I$ then $l(v_1 s_\alpha) > l(v_1)$ for every $s_\alpha \in W_{\Delta\setminus J}$. In particular, if $\alpha \in (\Delta \setminus J) \setminus (\Delta \setminus I)$ then we note that $v_1 s_\alpha \in W^{\Delta\setminus I}$ (for this see below [5]). Since $l(v_1 s_\alpha) > l(v_1)$ this is a contradiction to the assumption that $v_1$ is of maximal length in $W^{\Delta\setminus I}$.

Thus trivially $f_{v_1}^{\Delta\setminus I}$ is in the $R(\widetilde{T})^W$-span of $\{f_{v'}^{\Delta\setminus J} : v' \in C^J\}_{J\subseteq I}$. Now by a decreasing induction on $l(v)$ we can therefore assume that if $l(v_1) > l(v)$ then $f_{v_1}^{\Delta\setminus I}$ belongs to the $R(\widetilde{T})^W$-span of $\{f_{v'}^{\Delta\setminus J} : v' \in C^J\}_{J\subseteq I}$. Thus (1.11) and the induction assumption together imply (3) of Prop. 1.9. $\square$

**Lemma 1.10.** *For $I \subseteq \Delta$, let $\{f_v^{\Delta\setminus I} : v \in W^{\Delta\setminus I}\}$ denote the basis defined by Steinberg of $R(\widetilde{T})^{W_{\Delta\setminus I}}$ as an $R(\widetilde{T})^W$-module. Recall from (1.3) that $W^{\Delta\setminus I} = \bigsqcup_{J\subseteq I} C^J$. Then $\{f_v^{\Delta\setminus J} : v \in C^J\}_{J\subseteq I}$ also form a basis of $R(\widetilde{T})^{W_{\Delta\setminus I}}$ as $R(\widetilde{T})^W$-module. Moreover, if*

$$R(\widetilde{T})_I := \bigoplus_{v\in C^I} R(\widetilde{T})^W \cdot f_v^{\Delta\setminus I} \tag{1.12}$$

*then for every $I \subseteq \Delta$ we have the following direct sum decomposition as $R(\widetilde{T})^W$ modules:*

$$R(\widetilde{T})^{W_{\Delta\setminus I}} = \bigoplus_{J\subseteq I} R(\widetilde{T})_J. \tag{1.13}$$

[4] Since $l(x) = l(x') + l(y)$ and $x \in W^{\ell}_{\Delta\setminus J}(v)$, it follows that $x' \in W^{\ell}_{\Delta\setminus J}(v)$. Thus by (1.6) $p_v = p_{vx'}$. Hence $x^{-1}v^{-1}p_v = x_1^{-1}v^{-1}p_v \Leftrightarrow y^{-1}(vx')^{-1}p_{vx'} = y_1^{-1}(vx')^{-1}p_{vx'}$.

[5] For if $\beta \in \Delta \setminus I$, then $l(v_1 s_\alpha s_\beta) = l(v_1) + l(s_\alpha s_\beta) > l(v_1) + l(s_\alpha) = l(v_1 s_\alpha)$, since $s_\beta \neq s_\alpha$.

*This further implies that:*

$$R(\widetilde{T})^{W_{\Delta\setminus I}} = (\sum_{J\subsetneq I} R(\widetilde{T})^{W_{\Delta\setminus J}}) \bigoplus R(\widetilde{T})_I. \tag{1.14}$$

*Proof:* By Prop.1.9 (3) it follows that $\{f_v^{\Delta\setminus J} : v \in C^J\}_{J\subseteq I}$ span $R(\widetilde{T})^{W_{\Delta\setminus I}}$ as $R(\widetilde{T})^W$-module. It is not hard to see that $\{f_v^{\Delta\setminus J} : v \in C^J\}_{J\subseteq I}$ in fact form a basis of $R(\widetilde{T})^{W_{\Delta\setminus I}}$ as $R(\widetilde{T})^W$-module (for this see below [6]).

Since by (1.3) $W^{\Delta\setminus I} = \bigsqcup_{J\subseteq I} C^J$, we therefore have the following direct sum decomposition:

$$R(\widetilde{T})^{W_{\Delta\setminus I}} = \bigoplus_{J\subseteq I} \bigoplus_{v\in C^J} R(\widetilde{T})^W \cdot f_v^{\Delta\setminus J}.$$

Hence by (1.12) we further have:

$$R(\widetilde{T})^{W_{\Delta\setminus I}} = \bigoplus_{J\subseteq I} R(\widetilde{T})_J$$

for every $I \subseteq \Delta$. Now it follows by induction that

$$R(\widetilde{T})^{W_{\Delta\setminus I}} = (\sum_{J\subsetneq I} R(\widetilde{T})^{W_{\Delta\setminus J}}) \bigoplus R(\widetilde{T})_I.$$

□

**Remark 1.11.** In Lemma 1.10 we prove that the Steinberg basis elements for $R(\widetilde{T})^{W_{\Delta\setminus J}}$ which correspond to the indexing set $C^J \subseteq W^{\Delta\setminus I}$ for each $J \subseteq I$, together form another $R(\widetilde{T})^W$-basis for $R(\widetilde{T})^{W_{\Delta\setminus I}}$. The difference between the Steinberg basis for $R(\widetilde{T})^{W_{\Delta\setminus I}}$ and the new basis is the following: for all the elements of the Steinberg basis the superscript $\Delta \setminus I$ remains constant as the index $v$ in the subscript varies over the elements of $W^{\Delta\setminus I}$; whereas for the new basis the superscript varies with the index $v$ in the subscript. More explicitly, the superscript is $\Delta \setminus J$ whenever $v \in C^J$ where $W^{\Delta\setminus I} = \bigsqcup_{J\subseteq I} C^J$.

**Notation 1.12** *Henceforth throughout this paper we shall fix the following notation: whenever $v \in C^I$ we shall denote $f_v^{\Delta\setminus I}$ simply by $f_v$. We can drop the superscript in the notation without any ambiguity since $\{C^I : I \subseteq \Delta\}$ are disjoint. Therefore with the modified notation Lemma 1.10 implies that: $\{f_v : v \in W^{\Delta\setminus I} = \bigsqcup_{J\subseteq I} C^J\}$ form an $R(\widetilde{T})^W$-basis for $R(\widetilde{T})^{W_{\Delta\setminus I}}$ for every $I \subseteq \Delta$ and*

$$R(\widetilde{T})_I := \bigoplus_{v\in C^I} R(\widetilde{T})^W \cdot f_v$$

*satisfies (1.13) and (1.14).*

[6] This is because $R(\widetilde{G}) = R(\widetilde{T})^W$ is a domain and $R(\widetilde{T})^{W_{\Delta\setminus I}}$ is a free $R(\widetilde{T})^W$-module of rank $|W^{\Delta\setminus I}|$, it can be seen that $\{f_v^{\Delta\setminus J} : v \in C^J\}_{J\subseteq I}$ are linearly independent over $R(\widetilde{T})^W$ and hence form a basis of $R(\widetilde{T})^{W_{\Delta\setminus I}}$ as $R(\widetilde{T})^W$-module for every $I \subseteq \Delta$.

### 1.2.1 Comparison with Topological K-theory

Let $T_{comp} \subset T$ denote the maximal compact torus of $T$. Then any complex algebraic $T$-variety can be viewed as a topological $T_{comp}$-space. In particular, we have the algebraic $K$-group $K_T(X)$ and the topological $K$-group $K^{top}_{T_{comp}}(X)$. Now, since any algebraic vector bundle may be regarded as a topological vector bundle we have a natural homomorphism $K_T(X) \to K^{top}_{T_{comp}}(X)$ (see pp.272 [8]).

**Lemma 1.13.** *Let $X$ be a smooth projective variety on which $T$-acts with finitely many fixed points. Then the canonical map $K_T(X) \to K^{top}_{T_{comp}}(X)$ is an isomorphism.*

*Proof:* The lemma follows by Proposition 5.5.6 of [8] since $X \to pt$ is a $T$-equivariant cellular fibration and $K_T(pt) = R(T) \simeq R(T_{comp}) = K^{top}_{T_{comp}}(pt)$. $\square$

**Remark 1.14.** Let $G_{comp}$ be a maximal compact subgroup of $G$ such that $T_{comp} = G_{comp} \cap T$ is a maximal torus in $G_{comp}$. It has been proved in Theorem 4.4 of [18] that $K^{top}_{G_{comp}}(X) \simeq (K^{top}_{T_{comp}}(X))^W$. Now, since $K_T(X) \to K^{top}_{T_{comp}}(X)$ is $W$-invariant, it further follows from Theorem 1.8 that $K_G(X) \simeq K^{top}_{G_{comp}}(X)$.

# 2 K-theory of regular embeddings

We shall henceforth denote by $X$ a projective regular compactification of $G$. We follow the notations of §1.1 together with the following:

Let $\overline{T}$ denote the closure of $T$ in $X$. On $G$ the restriction of the action of $diag(T)$ is given by $(t,t) \cdot g = tgt^{-1}$ for all $g \in G$ and $t \in T$. This extends to an action on $X$. Thus $\overline{T}$ is an irreducible component of the fixed points of the torus $diag(T)$ and is therefore smooth (see Lemma 5.11.1. of [8]). Thus for the left action of $T$ (i.e. for the action of $T \times \{1\}$), $\overline{T}$ is a smooth complete toric variety.

We now recall certain facts and notations from §3.1 of [4] suitably adapted to the setting of $K$-theory.

By Prop. A1 of [4], $X^{T\times T}$ is contained in the union $X_c$ of all closed $G \times G$-orbits in $X$; moreover all such orbits are isomorphic to $G/B^- \times G/B$. Therefore by Theorem 2 of [23], $K_{T\times T}(X)$ embeds into $K_{T\times T}(X_c)$, the latter being a product of copies of the ring $K_{T\times T}(G/B^- \times G/B)$.

Let $\mathcal{F}$ be the fan associated to $\overline{T}$ in $X_*(T) \otimes \mathbb{R}$. Since $\overline{T}$ is complete, $\mathcal{F}$ is a subdivision of $X_*(T) \otimes \mathbb{R}$. Moreover, since $\overline{T}$ is invariant under $diag(W)$, the fan $\mathcal{F}$ is invariant under $W$, too. Since $X$ is a regular embedding, by Prop. A2

of [4], it follows that $\mathcal{F} = W\mathcal{F}_+$ where $\mathcal{F}_+$ is the subdivision of the positive Weyl chamber formed by the cones in $\mathcal{F}$ contained in this chamber. Therefore $\mathcal{F}$ is a smooth subdivision of the fan associated to the Weyl chambers, and the Weyl group $W$ acts on $\mathcal{F}$ by reflection about the Weyl chambers. Let $\overline{T}^+$ denote the toric variety associated to the fan $\mathcal{F}_+$. Let $\mathcal{F}(l)$ denote the set of maximal cones of $\mathcal{F}$. Then we know that $\mathcal{F}_+(l)$ parameterizes the closed $G \times G$-orbits in $X$. Hence $X^{T\times T}$ is parametrized by $\mathcal{F}_+(l) \times W \times W$. (The above facts follow from Prop. A1 and Prop. A2 of [4].)

For $\sigma \in \mathcal{F}_+(l)$, we denote by $Z_\sigma \simeq G/B^- \times G/B$ the corresponding closed orbit with base point $z_\sigma$, and by

$$\iota_\sigma : K_{T\times T}(X) \to K_{T\times T}(Z_\sigma) = K_{T\times T}(G/B^- \times G/B)$$

the restriction map. Moreover, for $f \in K_{T\times T}(Z_\sigma)$ and $u, v \in W$, we denote by $f_{u,v}$, the restriction of $f$ to the point $(u, v)z_\sigma$.

With the above notations, we have the following theorem. For the analogous result in the case of Chow ring see pp. 159 of [4].

**Theorem 2.1.** *For any projective regular embedding $X$ of $G$, the map*

$$\prod_{\sigma\in\mathcal{F}_+(l)} \iota_\sigma : K_{T\times T}(X) \to \prod_{\sigma\in\mathcal{F}_+(l)} K_{T\times T}(G/B^- \times G/B)$$

*is injective and its image consists in all families $(f_\sigma)$ $(\sigma \in \mathcal{F}_+(l))$ in $R(T) \otimes R(T)$, such that*

*(i) $f_{\sigma,us_\alpha,vs_\alpha} \equiv f_{\sigma,u,v} \pmod{1 - e^{-u(\alpha)} \otimes e^{-v(\alpha)}}$ whenever $\alpha \in \Delta$ and the cone $\sigma \in \mathcal{F}_+(l)$ has a facet orthogonal to $\alpha$, and that*

*(ii) $f_{\sigma,u,v} \equiv f_{\sigma',u,v} \pmod{1 - e^{-\chi}}$ whenever $\chi \in X^*(T)$ and the cones $\sigma$ and $\sigma' \in \mathcal{F}_+(l)$ have a common facet orthogonal to $\chi$.*

*(In (ii), $\chi$ is viewed as a character of $T \times T$ which is trivial on diag$(T)$ and hence is a character of $T$.)*

*Proof:* In the proof of the Theorem on pp.160 of [4] we have a complete description of all $T \times T$-invariant irreducible curves in $X$. We briefly recall here this description.

*$T \times T$-invariant curves in $X$:* Let $\gamma$ be a $T \times T$-invariant irreducible curve in $X$. Then $\gamma$ joins two $T \times T$-fixed points in $X$ and one of the following cases occur:

**(1)** $\gamma$ lies inside a closed orbit $Z_\sigma$. Thus by the description of $T$-invariant curves in $G/B$ (see §6.5 of [5]) it follows that $\gamma$ is conjugate in $W \times W$ to a curve $\gamma'$ joining $z_\sigma$ to $(s_\alpha, 1)z_\sigma$ or to $(1, s_\alpha)z_\sigma$ where $z_\sigma$ is the base point of $Z_\sigma$.

**(2)** $\gamma$ is conjugate in $W \times W$ to a curve $\gamma'$ joining the $T \times T$-fixed points $z_\sigma$ and $(s_\alpha, s_\alpha)z_\sigma$ of the closed orbit $Z_\sigma$, where $\gamma'$ is not contained in $Z_\sigma$. In this case the cone $\sigma$ in $\mathcal{F}_+(l)$ has a facet orthogonal to $\alpha$.

**(3)** $\gamma$ is conjugate in $W \times W$ to a projective line $\gamma'$ joining the $T \times T$-fixed points $z_\sigma$ and $z_{\sigma'}$ which are respectively the base points of distinct closed orbits $Z_\sigma$ and $Z_{\sigma'}$. In this case the cones $\sigma$ and $\sigma'$ in $\mathcal{F}_+(l)$ have a common facet.

In particular we observe that the set of $T \times T$-invariant irreducible curves in $X$ is finite.

Therefore by Theorem 1.3, the image of

$$\iota^* : K_{T\times T}(X) \to K_{T\times T}(X^{T\times T})$$

is defined by linear congruences $f_x \equiv f_y \pmod{(1 - e^{-\chi})}$ whenever $x, y \in X^{T\times T}$ are connected by a curve where $T \times T$ acts by the character $\chi$.

Further, observe that $T \times T$ acts on the curve joining $z_\sigma$ to $(s_\alpha, s_\alpha)z_\sigma$ by the character $(\alpha, \alpha)$, and on the curve joining $z_\sigma$ to $z_{\sigma'}$ by the character $\chi$ where $\sigma$ and $\sigma'$ have a common facet orthogonal to $\chi$. It therefore follows that the curves of type (1) define the image of $\prod_{\sigma\in\mathcal{F}_+} \iota_\sigma$, whereas curves of type (2) and (3) lead to congruences (i) and (ii). $\square$

**Corollary 2.2.** *The ring $K_{G\times G}(X)$ consists in all families $(f_\sigma)(\sigma \in \mathcal{F}_+(l))$ of elements of $R(T) \otimes R(T)$ such that:*

*(i)* $(s_\alpha, s_\alpha)f_\sigma \equiv f_\sigma \pmod{(1 - e^{-\alpha} \otimes e^{-\alpha})}$ *whenever $\alpha \in \Delta$ and the cone $\sigma \in \mathcal{F}_+(l)$ has a facet orthogonal to $\alpha$, and that*

*(ii)* $f_\sigma \equiv f_{\sigma'} \pmod{(1 - e^{-\chi})}$ *whenever $\chi \in X^*(T)$ and the cones $\sigma$ and $\sigma' \in \mathcal{F}_+(l)$ have a common facet orthogonal to $\chi$.*

*Proof:* By the isomorphism $(b)$ in §1.1, the ring $K_{G\times G}(G/B^- \times G/B)$ is isomorphic to $K_{T\times T}(G/B^- \times G/B))^{W\times W}$. It is further isomorphic to $R(T) \otimes R(T)$ via restriction to $z_\sigma$. Moreover, restriction of $f \in (K_{T\times T}(G/B^- \times G/B))^{W\times W} \simeq R(T) \otimes R(T)$ to $(u, v)z_\sigma$ is equal to $(u, v)f_\sigma$ where $f_\sigma$ denotes the restriction of $f$ to $z_\sigma$. So the relations (i) and (ii) of Theorem 2.1 reduce to (i) and (ii) of Cor.2.2. $\square$

We have the following relation between $K_{G\times G}(X)$ and $K_{T\times T}(\overline{T})$. This is analogous to the relation for semisimple adjoint groups and equivariant cohomology, due to Littelmann and Procesi (see [16]), and to the corresponding relation for the equivariant Chow ring of a regular group compactification due to Brion (see Cor.2 in §3.1 of [4]).

**Corollary 2.3.** *The inclusion $\overline{T} \hookrightarrow X$ induces the following isomorphisms:*

$$K_{G\times G}(X) \simeq K_{T\times T}(\overline{T})^W \simeq (K_T(\overline{T}) \otimes R(T))^W$$

*where the $W$-action on $K_{T\times T}(\overline{T})$ is induced from the action of $diag(W)$ on $\overline{T}$.*

*Proof:* Let $N$ be the normalizer of $T$ in $G$ and let $\overline{N}$ be its closure in $X$. Observe that $\overline{N}$ is the disjoint union of $(w,1)\overline{T}$ for $w\in W$. This can be seen as follows: We have $N=\bigcup_{w\in W} wT$. This implies that

$$\overline{N}=\bigcup_{w\in W}(w,1)\overline{T}.$$

Further, the map $y\mapsto (w,1)y \;\; \forall\, y\in\overline{T}$ is an isomorphism from $\overline{T}$ to $(w,1)\overline{T}$ on which the $T\times T$-action is twisted by $(w,1)$. In particular, the $T\times T$-fixed points in $(w,1)\overline{T}$ are $(w,1)\cdot\overline{T}^{T\times T}$.

Now, the set of fixed points $\overline{T}^{T\times T}$ is parametrized by $\mathcal{F}_{+}(l)\times diag(W)$. Therefore the set of $T\times T$-fixed points $(w,1)\cdot\overline{T}^{T\times T}$ is parametrized by $\mathcal{F}_{+}(l)\times(w,1)diag(W)$. However we know that $X^{T\times T}$ is parametrized by $\mathcal{F}_{+}(l)\times W\times W$ where $W\times W=\bigcup_{w\in W}(w,1)diag(W)$.

It follows that $(w,1)\overline{T}$ are disjoint, for otherwise the intersection of two of them should contain $T\times T$-fixed points which is a contradiction. Therefore we have:

$$\overline{N}=\bigsqcup_{w\in W}(w,1)\overline{T} \tag{2.1}$$

where for each $w\in W$, $(w,1)\overline{T}$ is an irreducible variety isomorphic to $\overline{T}$ with the appropriate twist for the $T\times T$-action.

In particular, $\overline{N}$ contains all fixed points of $T\times T$. It follows that restriction

$$K_{T\times T}(X)\to K_{T\times T}(\overline{N})$$

is injective.

Further, taking invariants of $K_{T\times T}(X)$ under $W\times W$, we see that the relations arising from curves of type (1) reduce to $(s_{\alpha},1)(f_{\sigma})\equiv f_{\sigma} \pmod{(1-e^{-\alpha})\otimes 1}$ or $(1,s_{\alpha})(f_{\sigma})\equiv f_{\sigma} \pmod{1\otimes(1-e^{-\alpha})}$ for $f_{\sigma}\in R(T)\otimes R(T)$ for every $\sigma\in\mathcal{F}_{+}(l)$.

However these relations trivially hold in $R(T)\otimes R(T)$, due to the fact that $s_{\alpha}(e^{\lambda})-e^{\lambda}$ is divisible in $R(T)$ by the element $1-e^{-\alpha}$ for every $\alpha\in\Delta$ and $\lambda\in\widetilde{\Lambda}$.

Therefore the non-trivial relations which describe the image of $K_{T\times T}(X)^{W\times W}$ arise from the curves of type (2) and (3).

From the description of $T\times T$-invariant curves in $X$ of type (2) and (3), it follows that any curve of type (2) or (3) lies in $(w,1)\overline{T}$ for a unique $w\in W$

(since any such curve is conjugate in $W \times W$ to a curve lying in $\overline{T}$). Thus $\overline{N}$ contains all $T \times T$-invariant curves which are not in any closed $(G \times G)$-orbit, that is curves of type (2) and (3). Thus we see that the restriction to $\overline{N}$ induces an isomorphism

$$K_{T\times T}(X)^{W\times W} \simeq K_{T\times T}(\overline{N})^{W\times W}.$$

Further, by (2.1) it follows that

$$K_{T\times T}(\overline{N}) \simeq \bigoplus_{w\in W} (w,1)K_{T\times T}(\overline{T})$$

where $(w,1)$ denotes the isomorphism on $K$-rings induced by the above isomorphism from $\overline{T}$ to $(w,1)\overline{T}$.

Thus the $W \times W$-module structure on $K_{T\times T}(\overline{N})$ is induced from the $diag(W)$-module structure on $K_{T\times T}(\overline{T})$. Thus we have

$$K_{T\times T}(\overline{N})^{W\times W} \simeq K_{T\times T}(\overline{T})^{W}.$$

Therefore we have the following isomorphisms

$$K_{T\times T}(X)^{W\times W} \simeq K_{T\times T}(\overline{N})^{W\times W} \simeq K_{T\times T}(\overline{T})^{W} \simeq (K_T(\overline{T}) \otimes R(T))^{W}.$$

(The last isomorphism is a consequence of the fact that we have a split exact sequence

$$1 \to diag(T) \to T \times T \to T \to 1$$

where the second map is $(t_1, t_2) \to t_1 t_2^{-1}$, and the splitting is given by $t \to (t, 1)$.

Thus $T \times T$ is canonically isomorphic to $diag(T) \times (T \times \{1\})$. Furthermore, by the definition of $T \times T$ action on $\overline{T}$ we see that $diag(T)$ acts trivially on $\overline{T}$. Therefore we have a ring isomorphism $K_{T\times T}(\overline{T}) \simeq R(diag(T)) \otimes K_T(\overline{T})$ (see 5.2.4 pp.244 of [8]). This isomorphism is further $W$-invariant since the $W$-action on the $K$-rings is induced from the action of $diag(W)$ on $\overline{T}$.) $\square$

**Remark 2.4.** Since $\mathcal{F}_+$ is a subdivision of the fan associated to the positive Weyl chamber, we have an induced proper morphism $\overline{T}^+ \to \mathbb{A}^l$. (Here $T$ acts linearly on $\mathbb{A}^l$ with weights being the simple roots.) Therefore, by the valuative criterion for properness it follows that $T$ acts on $\overline{T}^+$ with *enough limits* (see pp. 19 of [23] for the definition of action with enough limits). Thus by Cor. 5.11 and Cor. 5.12 of [23] it further follows that the restriction homomorphism

$$K_T(\overline{T}^+) \to \prod_{\sigma\in\mathcal{F}_+(l)} R(T_\sigma) = K_T((\overline{T}^+)^T)$$

is injective and an element $(a_\sigma) \in \prod_\sigma R(T_\sigma)$ is in the image of this homomorphism if and only if for any two *adjacent* maximal cones $\sigma$ and $\sigma'$, the restrictions of $a_\sigma$ and $a_{\sigma'}$ to $R(T_{\sigma\cap\sigma'})$ coincide, where $T_\tau \subseteq T$ denotes the stabilizer along the orbit $O_\tau$ for every $\tau \in \mathcal{F}_+$. Further, since $\overline{T}^+$ is smooth, and $T$ acts on $\overline{T}^+$ with finitely many fixed points and finitely many invariant curves, it can be seen (see Cor 5.11 of [23]) that Theorem 1.3 holds for $\overline{T}^+$. More precisely, the image of $K_T(\overline{T}^+)$ consists of elements $(a_\sigma) \in \prod_\sigma R(T_\sigma)$ such that $a_\sigma - a_{\sigma'} \equiv 0 \pmod{1 - e^{-\chi}}$ whenever $\sigma$ and $\sigma'$ have a common facet orthogonal to $\chi \in X^*(T)$.

The following proposition is a consequence of Corollary. 2.2.

**Proposition 2.5.** *We have the following chain of inclusions as $R(G) \otimes R(G)$-modules:*

$$R(T) \otimes R(G) \subseteq K_T(\overline{T}^+) \otimes R(G) \subseteq K_{G\times G}(X) \subseteq R(T)^{|\mathcal{F}_+(l)|} \otimes R(T).$$

*Moreover, $K_{G\times G}(X)$ is a module over $R(T) \otimes R(G)$.*

*Proof:* From the split exact sequence

$$1 \to diag(T) \to T \times T \to T \to 1$$

it follows that $R(T) \otimes R(T) \simeq R(T \times \{1\}) \otimes R(diag(T))$.

Recall that

$$\overline{N} = \bigsqcup_{w\in W} (w,1)\overline{T},$$

and any $T \times T$-invariant curve of type (2) or (3) in $X$ lies in $(w,1)\overline{T}$ for some $w \in W$. In particular, it follows that $diag(T)$ acts trivially on the curve $\gamma$ joining $(w,1)z_\sigma$ and $(w,1)(s_\alpha, s_\alpha)z_\sigma$ for every $\sigma \in \mathcal{F}_+(l)$ having a facet orthogonal to $\alpha \in \Delta$, and $w \in W$. Moreover, $T \times \{1\}$ acts on $\gamma$ by the character $\alpha$ (where the action is twisted by $(w,1)$ on the curves lying in $(w,1)\overline{T}$).

Similarly, $diag(T)$ acts trivially on the curve $\gamma$ joining $(w,1)z_\sigma$ and $(w,1)z_{\sigma'}$, and $T \times \{1\}$ acts on $\gamma$ by the character $\chi$, for all cones $\sigma$ and $\sigma' \in \mathcal{F}_+(l)$ having a common facet orthogonal to $\chi$.

Hence by Cor.2.2 it follows that $K_{G\times G}(X)$ consists in all families $(f_\sigma)(\sigma \in \mathcal{F}_+(l))$ of elements of $R(T \times \{1\}) \otimes R(diag(T)$ such that:

- *(i)* $(1, s_\alpha)f_\sigma(u,v) \equiv f_\sigma(u,v) \pmod{1 - e^{-\alpha(u)}}$ whenever $\alpha \in \Delta$ and the cone $\sigma \in \mathcal{F}_+(l)$ has a facet orthogonal to $\alpha$.
- *(ii)* $f_\sigma \equiv f_{\sigma'} \pmod{1 - e^{-\chi(u)}}$ whenever $\chi \in X^*(T)$ and the cones $\sigma$ and $\sigma' \in \mathcal{F}_+(l)$ have a common facet orthogonal to $\chi$.

where $u$ and $v$ denote the variables corresponding to $R(T\times\{1\})$ and $R(diag(T))$ respectively.

Now, by Remark 2.4 it follows that $K_T(\overline{T}^+)\otimes R(T)^W \subseteq \prod_{\sigma\in\mathcal{F}_+(l)} R(T_\sigma)\otimes R(T)$ is generated by the elements $(a_\sigma)\otimes b$, where $a_\sigma - a_{\sigma'} \equiv 0 \pmod{(1-e^{-\chi})}$, whenever $\sigma$ and $\sigma'$ share a facet orthogonal to $\chi\in X^*(T)$, and $b\in R(T)^W$.

Therefore, by identifying both $T\times\{1\}$ and $diag(T)$ naturally with $T$ keeping track of the ordering, we see that $K_T(\overline{T}^+)\otimes R(T)^W$ satisfies the relations $(i)$ and $(ii)$. Therefore it is a submodule of $K_{G\times G}(X)$. Moreover, since $K_T(\overline{T}^+)$ is an algebra over $R(T)$, it follows that $K_{G\times G}(X)$ is a module over $R(T)\otimes R(G)$. $\square$

We give below the example of wonderful compactification of $PGL(2,\mathbb{C})$. In particular we shall clearly see the curves of type (1) and (2) in this case.

**Example 2.6.** Let $G = PGL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm Id$. Then the projective space $\mathbb{P}(M(2,\mathbb{C}))$ is the wonderful compactification of $PGL(2,\mathbb{C})$, on which the action of $PGL(2,\mathbb{C})\times PGL(2,\mathbb{C})$ by multiplication on the left and on the right extends.

Let $E_{ij}$ denote the elementary matrix with 1 as $(i,j)th$ entry and 0 elsewhere for $1\le i,j\le 2$. In this case the Weyl group is $W=\{1=Id, s_\alpha=-E_{12}+E_{21}\}$, and $\overline{T}\simeq\mathbb{P}^1$ consists of the diagonal matrices in $\mathbb{P}(M(2,\mathbb{C}))$.

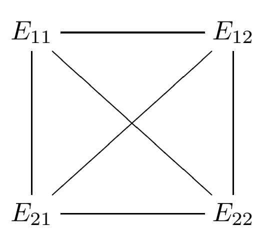
Further, the unique closed $PGL(2,\mathbb{C})\times PGL(2,\mathbb{C})$-orbit consists of the matrices of rank 1 in $\mathbb{P}(M(2,\mathbb{C}))$ and is isomorphic to $PGL(2,\mathbb{C})\times PGL(2,\mathbb{C})/(B^-\times B^+)$, choosing as base point the matrix $E_{11}$. Furthermore, $PGL(2,\mathbb{C})$ is the open orbit with base point $Id$.

The four $T\times T$ fixed points of $\mathbb{P}(M(2,\mathbb{C}))$ are: $E_{11}$, $E_{12}=(1,s_\alpha)E_{11}$, $E_{21}=(s_\alpha,1)E_{11}$ and $E_{22}=(s_\alpha,s_\alpha)E_{11}$. Further, the $T\times T$ curves are the following:

(1) $aE_{11}+bE_{12}$; $aE_{11}+bE_{21}$; $aE_{12}+bE_{22}$; $aE_{21}+bE_{22}$.

(2) $aE_{11}+bE_{22}$ and $aE_{12}+bE_{21}$.

where $aE_{ij}+bE_{pq}\ \ \forall\, a,b\in\mathbb{C}$, denotes the projective line joining $E_{ij}$ and $E_{pq}$ in $\mathbb{P}(M(2,\mathbb{C}))$ for $i,j,p,q\in\{1,2\}$. Pictorially we can view these curves as follows:



Thus we see that the curves of type (1) lie entirely in the unique closed orbit, whereas the curves of type (2) meet the open orbit.

Moreover, $\overline{N} = \overline{T} \sqcup (s_\alpha, 1)\overline{T}$ is the union of diagonal and the antidiagonal matrices. Hence $\overline{N}$ contains only the curves of type (2) and does not contain the curves of type (1).

In this case we do not have curves of type (3) since there is a unique closed $G \times G$-orbit.

**Remark 2.7.** Note that *all the results in this section hold analogously for* $K_{\widetilde{G}\times\widetilde{G}}(X)$ *and* $K_{\widetilde{T}\times\widetilde{T}}(X)$ where we take the natural actions of $\widetilde{G}\times\widetilde{G}$ and $\widetilde{T}\times\widetilde{T}$ through the canonical surjections to $G \times G$ and $T \times T$ respectively.

## 2.1 Determination of the structure of $K_{G\times G}(X)$

Let $X := \overline{G}$ be a projective regular embedding of $G$ and let $\overline{T}$ be the corresponding torus embedding.

Let $\mathcal{F}$ be the (smooth projective) fan associated to $\overline{T}$. Recall that the Weyl group $W$ acts on $\mathcal{F}$ by reflection about the Weyl chambers and the cones in $\mathcal{F}$ get permuted by this action of $W$, and each cone is stabilized by the reflections corresponding to the walls of the Weyl chambers on which it lies. Let $W_\tau$ denote the subgroup of $W$ which fixes the cone $\tau \in \mathcal{F}$. Then in particular, $W_\sigma = \{1\}\ \forall\ \sigma \in \mathcal{F}(l)$, and $W_{\{0\}} = W$.

Let $\{\rho_j : j = 1, \ldots d\}$ denote the set of edges of the fan $\mathcal{F}$ and let $\tau(1)$ denote the set of edges of the cone $\tau$ for every $\tau \in \mathcal{F}$. Let $v_j$ denote the primitive vector along the edge $\rho_j$. Let $O_\tau$ denote the $T$-orbit in $\overline{T}$ corresponding to $\tau \in \mathcal{F}$. Let $L_j$ denote the $T$-equivariant line bundle on $\overline{T}$ corresponding to the edge $\rho_j$. We note that, $L_j$ has a $T$-invariant section $s_j$ whose zero locus is $\overline{O_{\rho_j}}$. Recall that $\overline{T}^+$ denotes the toric variety associated to the fan $\mathcal{F}_+$.

Let $X_F := \prod_{\rho_j\in F}(1 - X_j)$ in the Laurent polynomial algebra $\mathbb{Z}[X_1^{\pm1}, \ldots X_d^{\pm1}]$, for every $F \subseteq \{\rho_j : j = 1, \ldots d\}$. In particular, let $X_\tau := X_{\tau(1)} = \prod_{\rho_j\in\tau(1)}(1 - X_j)$ for every $\tau \in \mathcal{F}$.

Recall from Theorem 6.4 of [23] we have the following Stanley-Reisner presentation of the $T$-equivariant $K$-ring of $\overline{T^+}$:

$$K_T(\overline{T}^+) \simeq \mathbb{Z}[X_j^{\pm1} : \rho_j \in \mathcal{F}_+(1)]/\langle X_F, \ \ \forall\ F \notin \mathcal{F}_+\rangle$$

where under the above isomorphism $X_j$ maps to $[L_j]$.

Further, we have the additive decomposition $K_T(\overline{T}^+) = \bigoplus_{\tau\in\mathcal{F}_+} C_\tau$, where

$$C_\tau := X_\tau \cdot \mathbb{Z}[X_j^{\pm1} : \rho_j \in \tau(1)].$$

Since we do not have an immediate reference for the above additive decomposition which may be well known, we give a proof of it in the following lemma.

**Lemma 2.8.** *We have the additive decomposition* $K_T(\overline{T}^+) = \bigoplus_{\tau\in\mathcal{F}_+} C_\tau$, *where*

$$C_\tau := X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)].$$

*Proof:* Let $\mathcal{I}$ be any finite indexing set. Then $\mathbb{Z}[X_j^{\pm 1} : j \in \mathcal{I}] = R[X_i^{\pm 1}] = R\bigoplus(1 - X_i)R[X_i^{\pm 1}]$ where $R := \mathbb{Z}[X_j^{\pm 1} : j \in \mathcal{I}, j \neq i]$. By induction on $|\mathcal{I}|$ we have the following direct sum decomposition:

$$\mathbb{Z}[X_j^{\pm 1} : j \in \mathcal{I}] = \bigoplus_{F\subseteq\mathcal{I}} X_F \cdot \mathbb{Z}[X_j^{\pm 1} : j \in F]$$

where $X_F := \prod_{j\in F}(1 - X_j)$.

Now, let $F \subseteq \mathcal{I}$ and $i \notin F$. Then we have $X_F \cdot \mathbb{Z}[X_j^{\pm 1} : j \in \mathcal{I}] = X_F \cdot R[X_i^{\pm 1}] = X_F \cdot R \bigoplus X_{F'} \cdot R[X_i^{\pm 1}]$ where $R := \mathbb{Z}[X_j^{\pm 1} : j \in \mathcal{I}, j \neq i]$ and $X_{F'} = X_F(1 - X_i)$. Thus by induction on $|\{i : i \notin F\}|$ it follows that we have the following direct sum decomposition:

$$X_F \cdot \mathbb{Z}[X_j^{\pm 1} : j \in \mathcal{I}] = \bigoplus_{F\subseteq F'} X_{F'} \cdot \mathbb{Z}[X_j^{\pm 1} : j \in F'].$$

In particular, applying the above arguments for $\mathcal{I} = \mathcal{F}_+(1)$ we get:

$$\mathbb{Z}[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)] = \bigoplus_{F\subseteq\mathcal{F}_+(1)} X_F \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in F]$$

and further for $F \subseteq \mathcal{F}_+(1)$ such that $F \notin \mathcal{F}_+$ we get:

$$X_F \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)] = \bigoplus_{F\subseteq F'} X_{F'} \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in F']$$

Since $F \subseteq F'$ implies $F' \notin \mathcal{F}_+$ it follows that we have the following direct sum decomposition:

$$\langle X_F, \ \ \forall\ F \notin \mathcal{F}_+\rangle = \bigoplus_{F\notin\mathcal{F}_+} X_F \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in F]$$

The lemma now follows by the Stanley-Reisner presentation of $K_T(\overline{T}^+)$. □

**Remark 2.9.** Note that although we state Lemma 2.8 for $\overline{T}^+$ it is not hard to see that an analogous additive decomposition holds for the $T$-equivariant $K$-ring of any smooth $T$-toric variety.

Whenever $\sigma, \tau \in \mathcal{F}$ (resp. $\mathcal{F}_+$) span a cone in $\mathcal{F}$ (resp. $\mathcal{F}_+$), we shall denote the cone spanned by them as $\gamma := \langle\tau, \sigma\rangle$.

**Theorem 2.10.** *Let $X := \overline{G}$ be a projective regular embedding of $G$ and let $\overline{T}$ be the corresponding torus embedding. Then, $K_{G\times G}(X)$ has the following direct sum decomposition as $1 \otimes R(G)$-module:*

$$K_{G\times G}(X) \simeq \bigoplus_{\tau\in\mathcal{F}_+} C_\tau \otimes R(T)^{W_\tau}$$

*where $R(G) = R(T)^W$ acts naturally on the second factor in each piece of the above decomposition. Further, the* multiplicative structure *of $K_{G\times G}(X)$ can be described from the above decomposition as follows: Let $a_\tau \otimes b_\tau \in C_\tau \otimes R(T)^{W_\tau}$ and $a_\sigma \otimes b_\sigma \in C_\sigma \otimes R(T)^{W_\sigma}$. Then*

$$(a_\tau\otimes b_\tau)\cdot(a_\sigma\otimes b_\sigma) = \begin{cases} a_\tau \cdot a_\sigma \otimes b_\tau \cdot b_\sigma, & \text{if } \tau \text{ and } \sigma \text{ span the cone } \gamma \\ 0 & \text{if } \tau \text{ and } \sigma \text{ do not span a cone in } \mathcal{F}_+ \end{cases}$$

(Note that $a_\tau \cdot a_\sigma \otimes b_\tau \cdot b_\sigma \in C_\gamma \otimes R(T)^{W_\gamma}$, and the multiplication in the first factor is as in $K_T(\overline{T}^+)$ where $C_\tau \cdot C_\sigma \subseteq C_\gamma$.)

*Proof:* We have the following isomorphisms by Cor.2.3:

$$K_{G\times G}(X) \simeq K_{T\times T}(\overline{T})^W \simeq (K_T(\overline{T}) \otimes R(T))^W$$

Now by Theorem 6.4 of [23] we have the following Stanley-Reisner presentation of the $T$-equivariant $K$-ring of $\overline{T}$:

$$K_T(\overline{T}) \simeq \mathbb{Z}[X_1^{\pm 1}, \dots X_d^{\pm 1}]/\langle X_F, \ \ \forall\ F \notin \mathcal{F}\rangle \tag{2.2}$$

Since $W$ acts on $\mathcal{F}$, we have an action of $W$ on $\mathbb{Z}[X_1^{\pm 1}, \dots X_d^{\pm 1}]$, given by $w(X_{\rho_j}^{\pm 1}) = X_{w(\rho_j)}^{\pm 1}$ for every $w \in W$. Therefore, $w(X_F) = X_{w(F)}$ for $F \subseteq \{\rho_j : j = 1, \dots d\}$ and $w \in W$, and since $W$ permutes the cones of $\mathcal{F}$ we further get an action of $W$ on the Stanley-Reisner algebra $\mathbb{Z}[X_1^{\pm 1}, \dots X_d^{\pm 1}]/\langle X_F\ \forall\ F \notin \mathcal{F}\rangle$. The above isomorphism is an isomorphism of $W$-modules, where the $W$-action on $K_T(\overline{T})$ is induced by the $diag(W)$-action on $\overline{T}$.

Further, (see Lemma 2.8 and Remark 2.9)

$$\mathbb{Z}[X_1^{\pm 1}, \dots X_d^{\pm 1}]/\langle X_F, \ \ \forall\ F \notin \mathcal{F}\rangle = \bigoplus_{\tau\in\mathcal{F}} X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)]$$

where we have the natural action of $W$ on the right hand side given by:

$$w \cdot (X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)]) = X_{w(\tau)} \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in w(\tau)(1)], \quad \forall\ \ w \in W.$$

Therefore we have:

$$K_T(\overline{T}) = \bigoplus_{\tau\in\mathcal{F}_+} \bigoplus_{w\in W/W_\tau} X_{w(\tau)} \cdot \mathbb{Z}[w(X_j^{\pm 1}) : \rho_j \in \tau(1)]$$

Here $W_\tau$ denotes the subgroup of $W$ which fixes the cone $\tau \in \mathcal{F}_+$. Hence we have as $W$-modules:

$$K_T(\overline{T}) = \bigoplus_{\tau \in \mathcal{F}_+} Ind_{W_\tau}^W C_\tau$$

where, $C_\tau := X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)]$.

Further, since $C_\tau$ is fixed by $W_\tau$, hence $Ind_{W_\tau}^W C_\tau \simeq \mathbb{Z}[W/W_\tau] \otimes C_\tau$.

Thus we have:

$$K_T(\overline{T}) \otimes R(T) = \bigoplus_{\tau \in \mathcal{F}_+} \mathbb{Z}[W/W_\tau] \otimes C_\tau \otimes R(T).$$

Now by taking $W$-invariants on either side we get:

$$(K_T(\overline{T}) \otimes R(T))^W = \bigoplus_{\tau \in \mathcal{F}_+} C_\tau \otimes R(T)^{W_\tau}. \tag{2.3}$$

Thus we get the following additive decomposition:

$$K_{G\times G}(X) \simeq \bigoplus_{\tau \in \mathcal{F}_+} C_\tau \otimes R(T)^{W_\tau}.$$

We shall now describe the multiplication on the right hand side of the above decomposition which will make the above isomorphism a ring isomorphism.

First we note that the isomorphism $K_{G\times G}(X) \simeq (K_T(\overline{T}) \otimes R(T))^W$ is a ring isomorphism and hence preserves the multiplicative structure.

Further, since (2.2) is a ring isomorphism the multiplication in $K_T(\overline{T})$ is determined by multiplication in the Stanley-Reisner ring $\mathbb{Z}[X_1^{\pm 1}, \ldots X_d^{\pm 1}]/\langle X_F, \ \ \forall\ F \notin \mathcal{F}\rangle$ (see §6.2 of [23]). Moreover, the multiplication in

$$\mathbb{Z}[X_1^{\pm 1}, \ldots X_d^{\pm 1}]/\langle X_F, \ \ \forall\ F \notin \mathcal{F}\rangle = \bigoplus_{\tau \in \mathcal{F}} C_\tau$$

is determined by the products $C_\tau \cdot C_\sigma$ for $\tau, \sigma \in \mathcal{F}$, where $C_\tau \cdot C_\sigma \subseteq C_\gamma$ for $\gamma = \langle \tau, \sigma \rangle$. Similarly, the multiplication in $K_T(\overline{T}^+) = \bigoplus_{\tau \in \mathcal{F}_+} C_\tau$ is defined by multiplying $C_\tau$ and $C_\sigma$ for $\tau$ and $\sigma$ in $\mathcal{F}_+$. Moreover, if $\tau$ and $\sigma$ span a cone in $\mathcal{F}_+$ then $C_\tau \cdot C_\sigma \subseteq C_\gamma$ where $\gamma = \langle \sigma, \tau \rangle$, and if $\tau$ and $\sigma$ do not span any cone in $\mathcal{F}_+$ then $C_\tau \cdot C_\sigma = 0$.

Furthermore, the multiplicative structure on $K_T(\overline{T}) = \bigoplus_{\tau \in \mathcal{F}_+} Ind_{W_\tau}^W C_\tau$ is induced from the multiplication in $K_T(\overline{T}^+) = \bigoplus_{\tau \in \mathcal{F}_+} C_\tau$. This is because, for $w', w'' \in W$, $w'(\sigma)$ and $w''(\tau)$ span a cone in $\mathcal{F}$ if and only if $\tau$ and $\sigma$ span a cone in $\mathcal{F}_+$, and there exists $w \in W$ such that $w(\sigma) = w'(\sigma)$ and $w(\tau) = w''(\tau)$.

Moreover, if $\tau$ and $\sigma$ span $\gamma$ in $\mathcal{F}_+$, $w(\tau)$ and $w(\sigma)$ span $w(\gamma)$ in $w(\mathcal{F}_+)$ for every $w \in W$.

In particular, let $g_\sigma = w'(f_\sigma) \in C_{w'(\sigma)}$ and $g_\tau = w''(f_\tau) \in C_{w''(\tau)}$, where $f_\sigma \in C_\sigma$ and $f_\tau \in C_\tau$ for $\tau, \sigma \in \mathcal{F}_+$. Then $g_\sigma \cdot g_\tau = 0$ if $\sigma$ and $\tau$ do not span a cone in $\mathcal{F}_+$, or if there does not exist any $w \in W$ such that $w(\sigma) = w'(\sigma)$ and $w(\tau) = w''(\tau)$. Otherwise $g_\sigma \cdot g_\tau = w(f_\sigma \cdot f_\tau) \in C_{w(\gamma)}$, where $\gamma = \langle \tau, \sigma \rangle$, and $w(\sigma) = w'(\sigma)$ and $w(\tau) = w''(\tau)$ for $w \in W$.

Further note that, whenever $\gamma = \langle \tau, \sigma \rangle$ in $\mathcal{F}_+$ we have the product: $R(T)^{W_\tau} \cdot R(T)^{W_\sigma} \subseteq R(T)^{W_\gamma}$, where $R(T)^{W_\tau}$ and $R(T)^{W_\sigma}$ are both subrings of $R(T)^{W_\gamma}$.

Thus we see that the identity (2.3) above induces a multiplicative isomorphism, where the multiplication in $\bigoplus_{\tau \in \mathcal{F}_+} C_\tau \otimes R(T)^{W_\tau}$ is as described in the statement of the theorem. $\square$

**Corollary 2.11.** *The ring $K_{G\times G}(X) \simeq \bigoplus_{\tau \in \mathcal{F}_+} C_\tau \otimes R(T)^{W_\tau}$ admits a multi-filtration $\{F_\tau\}_{\tau \in \mathcal{F}_+}$, where the filtered pieces are*

$$F_\tau = \bigoplus_{\tau \prec \sigma} C_\sigma \otimes R(T)^{W_\sigma},$$

*where $F_\tau \supseteq F_\sigma$ whenever $\tau \prec \sigma$, and $F_{\{0\}} = K_{G\times G}(X)$. Further, under the multiplication described in Theorem 2.10, we have $F_\tau \cdot F_\sigma \subseteq F_\gamma$ where $\gamma = \langle \tau, \sigma \rangle$. In particular, $F_{\{0\}} \cdot F_\tau \subseteq F_\tau$ for all $\tau \in \mathcal{F}_+$. Moreover, since $K_T(\overline{T}^+) \otimes R(T)^W \subseteq F_{\{0\}}$, it follows that $K_{G\times G}(X)$ is a module over $K_T(\overline{T}^+) \otimes R(T)^W$ and each filtered piece $F_\tau$ is a $K_T(\overline{T}^+) \otimes R(T)^W$-submodule. Furthermore, the $K_T(\overline{T}^+) \otimes R(T)^W$-module structure on $K_{G\times G}(X)$ given by the above decomposition is compatible with the canonical $R(T) \otimes R(T)^W$-module structure on $K_{G\times G}(X)$ coming from the inclusion in Prop. 2.5.*

*Proof:* The existence of the filtration $\{F_\tau\}_{\tau \in \mathcal{F}_+}$ follows by definition. Further, the filtered pieces multiply by the multiplication rule defined in Theorem 2.10 and hence it follows that: $F_\tau \cdot F_\sigma \subseteq F_\gamma$ whenever $\gamma = \langle \tau, \sigma \rangle$, and $F_\tau \cdot F_\sigma = 0$ whenever $\tau$ and $\sigma$ do not span a cone in $\mathcal{F}_+$.

Recall by Prop. 2.5 that we have an inclusion $K_T(\overline{T}^+) \otimes R(T)^W \subseteq K_{G\times G}(X)$ as a subring, which further gives $K_{G\times G}(X)$ a canonical $R(T) \otimes R(T)^W$-module structure. Now, under the above isomorphism, $K_T(\overline{T}^+) \otimes R(T)^W$ maps to $\bigoplus_{\tau \in \mathcal{F}_+} C_\tau \otimes R(T)^W \subseteq F_{\{0\}}$. Since $F_{\{0\}} \cdot F_\tau \subseteq F_\tau$, it follows that each $F_\tau$ for $\tau \in \mathcal{F}_+$ is a module over $K_T(\overline{T}^+) \otimes R(T)^W$. Moreover, the decomposition $K_{G\times G}(X) \simeq \bigoplus_{\tau \in \mathcal{F}_+} C_\tau \otimes R(T)^{W_\tau}$, preserves the multiplicative structure. Thus the above defined $R(T) \otimes R(T)^W \subseteq K_T(\overline{T}^+) \otimes R(T)^W$-module structure on $K_{G\times G}(X) \simeq F_{\{0\}}$, is compatible with the canonical structure given in Prop. 2.5. $\square$

The following corollary can be thought of as a geometric reinterpretation of Theorem 2.10.

**Corollary 2.12.** *Let $N_\tau \simeq \bigoplus_{\rho_j \in \tau(1)} L_j$ be the normal bundle of $V_\tau = \overline{O_\tau}$ in $\overline{T}$. Let $N_\tau \mid_{O_\tau}$ denote the restriction of the normal bundle to $O_\tau$ so that*

$$\lambda_{-1}(N_\tau \mid_{O_\tau}) := \prod_{\rho_j \in \tau(1)} (1 - [L_j] \mid_{O_\tau}) \in K_T(O_\tau).$$

*Then we have the following decomposition:*

$$K_{G\times G}(X) \simeq \bigoplus_{\tau \in \mathcal{F}_+} \lambda_{-1}(N_\tau \mid_{O_\tau}) \cdot K_T(O_\tau) \otimes R(T)^{W_\tau}.$$

*Let $P_\tau := \lambda_{-1}(N_\tau \mid_{O_\tau}) \cdot K_T(O_\tau) \otimes R(T)^{W_\tau}$ for each $\tau \in \mathcal{F}_+$. Then the above decomposition is a ring isomorphism where the* multiplication *on the right hand side is given as follows:*

$$P_\tau \cdot P_\sigma \subseteq \begin{cases} P_\gamma & \text{if } \tau \text{ and } \sigma \text{ span the cone } \gamma \text{ in } \mathcal{F}_+ \\ 0 & \text{if } \tau \text{ and } \sigma \text{ do not span a cone in } \mathcal{F}_+ \end{cases}$$

*Proof:* Observe that $\mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)] \simeq K_T(O_\tau)$ since $K_T(O_\tau) = K_T(T/T_\tau) = R(T_\tau)$ where, $T_\tau$ is the stabilizer of the orbit $O_\tau$. Indeed this isomorphism is induced from the map (6.2) pp. 27 of [23] composed with the restriction to $R(T_\tau)$, and is hence compatible with the isomorphism (2.2) above. Note that under the above isomorphism $X_j$ maps to $[L_j \mid_{O_\tau}]$.

Thus $C_\tau := X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)] \simeq \lambda_{-1}(N_\tau \mid_{O_\tau}) \cdot K_T(O_\tau) = P_\tau$, where $N_\tau \mid_{O_\tau} \simeq \bigoplus_{\rho_j \in \tau(1)} L_j \mid_{O_\tau}$ denotes the restriction of the normal bundle of $V_\tau$ to $O_\tau$. Therefore, substituting the above isomorphisms in (2.3) we have the following additive decomposition:

$$(K_T(\overline{T}) \otimes R(T))^W \simeq \bigoplus_{\tau \in \mathcal{F}_+} \lambda_{-1}(N_\tau \mid_{O_\tau}) \cdot K_T(O_\tau) \otimes R(T)^{W_\tau}.$$

Since by Cor.2.3 we have $K_{G\times G}(X) \simeq (K_T(\overline{T}) \otimes R(T))^W$, we get the required decomposition of $K_{G\times G}(X)$.

Further, since the torus embedding $\overline{T}$ is regular, the isotropy group $T_\tau$ has a dense orbit in the normal space $N(x)$ to $O_\tau$ at $x \in O_\tau$. In particular, this implies that the eigenspace of $N(x)$ corresponding to the trivial character of $T_\tau$ is zero. Thus by Lemma 4.2 of [23] it follows that $\lambda_{-1}(N_\tau \mid_{O_\tau})$ is not a zero divisor in $K_T(O_\tau)$. Thus we see that each piece $\lambda_{-1}(N_\tau \mid_{O_\tau}) \cdot K_T(O_\tau) \otimes R(T)^{W_\tau}$ is isomorphic to $K_T(O_\tau) \otimes R(T)^{W_\tau}$ for every $\tau \in \mathcal{F}_+$.

Furthermore, since $\gamma = \langle \tau, \sigma \rangle$, we have the restriction maps $R(T_\gamma) \to R(T_\tau)$ $R(T_\gamma) \to R(T_\sigma)$ induced by the canonical inclusions $T_\tau \subseteq T_\gamma$ and $T_\sigma \subseteq T_\gamma$. These restriction maps further admit splittings which are respectively given by $[L_j] \mid_{O_\tau} \mapsto [L_j] \mid_{O_\gamma}$ for $\rho_j \in \tau(1)$ and $[L_j] \mid_{O_\sigma} \mapsto [L_j] \mid_{O_\gamma}$ for $\rho_j \in \sigma(1)$.

In particular, $\lambda_{-1}(N_\tau \mid_{O_\tau}) = \prod_{\rho_j \in \tau(1)} (1 - [L_j] \mid_{O_\tau}) \in K_T(O_\tau)$ and $\lambda_{-1}(N_\sigma \mid_{O_\sigma}) = \prod_{\rho_j \in \sigma(1)} (1 - [L_j] \mid_{O_\sigma}) \in K_T(O_\sigma)$ multiply as elements in

$K_T(O_\gamma)$ to give

$$\prod_{\rho_j\in\tau(1)}(1-[L_j]\mid_{O_\gamma})\cdot\prod_{\rho_j\in\sigma(1)}(1-[L_j]\mid_{O_\gamma})=\prod_{\rho_j\in\gamma(1)}(1-[L_j]\mid_{O_\gamma})\cdot\prod_{\rho_j\in(\tau\cap\sigma)(1)}(1-[L_j]\mid_{O_\gamma}).$$

Thus the right hand side is divisible by $\lambda_{-1}(N_\gamma\mid_{O_\gamma})=\prod_{\rho_j\in\gamma(1)}(1-[L_j]\mid_{O_\gamma}) \in K_T(O_\gamma)$.

Now, by defining multiplication on the right hand side as in Theorem 2.10 the corollary follows. Thus the above decomposition of $K_{G\times G}(X)$ is a ring isomorphism. □

The structure of rational equivariant cohomology of regular embeddings has been described in complete detail in [2]. However for comparison with the setting of $K$-theory, we give below the analogous statement in the case of cohomology which we obtain by proceeding along similar steps as in Theorem 2.10.

We follow the notations in the beginning of this section except for the following modifications: Let $X_F:=\prod_{\rho_j\in F}X_j$ for every $F\subseteq\{\rho_j : j=1,\dots d\}$ in the polynomial algebra $\mathbb{Q}[X_1,\dots X_d]$. In particular, let $X_\tau:=X_{\tau(1)}=\prod_{\rho_j\in\tau(1)}X_j$ for every $\tau\in\mathcal{F}$. Let $S:=H_T^*(pt)$ be the symmetric algebra over $\mathbb{Q}$ of $X^*(T)$. By Theorem 8, pp. 7 of [2] we know that:

$$H_T^*(\overline{T})\simeq\mathbb{Q}[X_1,\dots X_d]/\langle X_F\ \forall\ F\notin\mathcal{F}\rangle$$

Let $e(N_\tau)$ denote the equivariant Euler class of the normal bundle of $V(\tau)=\overline{O_\tau}$ which is equal to the top chern class of $\bigoplus_{\rho_j\in\tau(1)}L_j$. We then have the following (also see Theorem 2.3 of [16]) for equivariant cohomology of the wonderful compactification of semisimple adjoint groups, and the corresponding result for Chow ring of a regular compactification of a connected reductive group in Cor. 2, p.161 of [4]):

**Theorem 2.13.**

$$H^*_{G\times G}(X)\simeq(H_T^*(\overline{T})\otimes S)^W\simeq\bigoplus_{\tau\in\mathcal{F}_+}e(N_\tau\mid_{O_\tau})\cdot H_T^*(O_\tau)\otimes S^{W_\tau}$$

**Remark 2.14.** As observed in page 3 of [23] the above results on algebraic and topological K-theory of $X$ hold with integral coefficients.

### 2.1.1 Application to Ordinary K-theory

In this section we shall consider $K_{\widetilde{G}\times\widetilde{G}}(X)$, in view of applying Theorem 4.2 of [19] to obtain the results for the ordinary $K$-ring of $X$. Moreover, by Remark 2.7 we can apply the contents of §2 to $K_{\widetilde{G}\times\widetilde{G}}(X)$ and $K_{\widetilde{T}\times\widetilde{T}}(X)$.

**Proposition 2.15.** *Consider the principal $(B^-\times B)$- bundle $G\times G\to G/B^-\times G/B$. Further, we have a canonical action of $B^-\times B$ on $\overline{T}$ through the surjection*

$B^- \times B \to T \times T$. *We consider the associated bundle* $(G \times G) \times_{B^- \times B} \overline{T}$ *which is a toric bundle with fibre* $\overline{T}$ *over* $G/B^- \times G/B$. *We then have the following description of* $K(X)$*:*

$$K(X) \simeq K((G \times G) \times_{B^- \times B} \overline{T})^{diag(W)}.$$

*Proof:* By Cor.2.3 we have: $K_{\widetilde{G} \times \widetilde{G}}(X) \simeq K_{\widetilde{T} \times \widetilde{T}}(\overline{T})^{diag(W)}$. Now observe that $K_{\widetilde{B}^- \times \widetilde{B}}(\overline{T}) = K_{\widetilde{G} \times \widetilde{G}}((\widetilde{G} \times \widetilde{G}) \times_{\widetilde{B}^- \times \widetilde{B}} \overline{T})$ (see 5.2.17 of [8]). Further, the restriction homomorphism $K_{\widetilde{B}^- \times \widetilde{B}}(\overline{T}) \to K_{\widetilde{T} \times \widetilde{T}}(\overline{T})$ is an isomorphism (see 5.2.18 of [8]). Thus we get the following isomorphism:

$$K_{\widetilde{G} \times \widetilde{G}}(X) \simeq K_{\widetilde{G} \times \widetilde{G}}((\widetilde{G} \times \widetilde{G}) \times_{\widetilde{B}^- \times \widetilde{B}} \overline{T})^{diag(W)}.$$

Note that both sides of the above isomorphism are $R(\widetilde{G}) \otimes R(\widetilde{G})$-algebras and further, the above isomorphism is an isomorphism as $R(\widetilde{G}) \otimes R(\widetilde{G})$-algebras (here we use the fact that $R(\widetilde{G}) \otimes R(\widetilde{G})$ is invariant under the action of $diag(W)$). Now, applying the isomorphism $(c)$ of §1.2 for $\widetilde{G} \times \widetilde{G}$ we get:

$$K(X) \simeq K((\widetilde{G} \times \widetilde{G}) \times_{\widetilde{B}^- \times \widetilde{B}} \overline{T})^{diag(W)}.$$

Further, since the relative $T$-embedding $(\widetilde{G} \times \widetilde{G}) \times_{\widetilde{B}^- \times \widetilde{B}} \overline{T}$, where $\widetilde{B}^- \times \widetilde{B}$ acts on the fibre $\overline{T}$ via the surjection to $B^- \times B$, is canonically isomorphic to $(G \times G) \times_{B^- \times B} \overline{T}$ we have the proposition. □

**Remark 2.16.** Recall that we have the Cartan decomposition $X = G_{comp} \overline{T} G_{comp}$ (see [11, pp. 585]) where $G_{comp}$ is a maximal compact subgroup of $G$ such that $T_{comp} = T \cap G_{comp}$ is a maximal compact torus in $G_{comp}$. Hence for the topological $K$-theory we have the following isomorphism

$$K^{top}(X) \simeq K^{top}((G_{comp} \times G_{comp}) \times_{T_{comp} \times T_{comp}} \overline{T})^{diag(W)},$$

which is obtained via pullback through the canonical map $(G_{comp} \times G_{comp}) \times_{T_{comp} \times T_{comp}} \overline{T} \to X$ (see [11, pp.585-588]).

**Remark 2.17.** The above description of $K(X)$ is analogous to the description of $H^*(X; \mathbb{Q})$ in Theorem (2.2) of [11] in the case when $X$ is the wonderful compactification. Also see [21] for the computation of the Grothendieck ring of a relative torus embedding over an arbitrary base, analogous to the computation of cohomology in §3 of [11].

**Remark 2.18.** Here we mention that for the case of a smooth complete toric variety the structure of the $T$-equivariant and ordinary $K$-theory is well known (see §6 of [23] for the computation of equivariant and ordinary $K$-theory of any smooth toric variety, and also see [21] for the ordinary $K$-theory of a smooth complete toric variety using different methods).

## 3 K-theory of the wonderful compactification

In this section $X := \overline{G_{ad}}$ the wonderful compactification of the semisimple adjoint group $G_{ad}$.

It follows from (1.1) that $G^{ss}$ is the universal cover of $G_{ad}$, and $T_{ad} := T^{ss}/C$ is the maximal torus of $G_{ad}$. Recall that $rank\ (G_{ad}) = rank\ (G^{ss}) = r$ which is the semisimple rank of $G$.

The toric variety $\overline{T_{ad}}$ then corresponds to the fan $\mathcal{F}_{ad}$ in $X_*(T_{ad})\otimes\mathbb{R}$, which is the fan associated to the Weyl chambers. Moreover, $\overline{T_{ad}}^{+} \simeq \mathbb{A}^r$, where $T_{ad}$ acts on $\mathbb{A}^r$ by the embedding $t \mapsto (t^{\alpha_1}, \ldots, t^{\alpha_r})$. Thus $(\mathcal{F}_{ad})_+$ is the fan associated to the positive Weyl chamber $\mathcal{C}^+$ where the edges of $(\mathcal{F}_{ad})_+$ are generated by the fundamental coweights $\omega_1^\vee, \ldots, \omega_r^\vee$ dual to the simple roots $\alpha_1, \ldots, \alpha_r$.

**Notation 3.1** *Henceforth in this section we let: $G := G^{ss}$, a semisimple simply connected algebraic group, $B := B^{ss}$ a Borel subgroup and $T := T^{ss}$ a maximal torus of $G$.*

Following Remark 2.7 we shall consider $G \times G$ and $T \times T$-equivariant $K$-theory of $X$, where we take the natural actions of $G\times G$ and $T\times T$ on $X$ through the canonical surjections to $G_{ad}\times G_{ad}$ and $T_{ad}\times T_{ad}$ respectively. In particular, we shall apply the contents of §1 and §2 for $K_{G\times G}(X)$ and $K_{T\times T}(X)$.

As in the proof of Prop. 2.5, we denote by $u$ and $v$ the first and second variables of $R(T) \times R(T)$ respectively.

**Lemma 3.2.** *The ring $K_{G\times G}(X) \subseteq R(T)\otimes R(T)$ consists of elements $f(u,v) \in R(T) \otimes R(T)$ that satisfy the relations $(1, s_\alpha)f(u,v) \equiv f(u,v) \pmod{1 - e^{-\alpha(u)}}$ for every $\alpha \in \Delta$.*

*Proof:* This follows immediately from the proof of Prop. 2.5 since there is only one maximal dimensional cone in $(\mathcal{F}_{ad})_+$ which has a facet orthogonal to $\alpha$ for every $\alpha \in \Delta$. Thus in this case there are no relations of type $(ii)$ and the relations of type $(i)$ are as given above. $\square$

It follows from the above lemma that $R(T) \otimes R(G) = R(T) \otimes R(T)^W \subseteq K_{G\times G}(X)$ as a subring. In particular, $K_{G\times G}(X)$ is a module over $R(T)\otimes R(G)$, and the following theorem describes explicitly this module structure.

**Theorem 3.3.** *The ring $K_{G\times G}(X)$ has the following direct sum decomposition as $R(T) \otimes R(G)$-module:*

$$K_{G\times G}(X) = \bigoplus_{I\subseteq\Delta} \prod_{\alpha\in I} (1 - e^{-\alpha(u)}) R(T) \otimes R(T)_I.$$

*Further, the above direct sum is a free $R(T) \otimes R(G)$-module of rank $|W|$ with basis*

$$\{\prod_{\alpha \in I}(1 - e^{-\alpha(u)}) \otimes f_v : \ v \in C^I \text{ and } I \subseteq \Delta\},$$

*where $C^I$ is as defined in* (1.3) *and $\{f_v\}$ is as in Notation 1.12.*

*Proof :* Recall from Lemma 1.10 that we have the following decompositions as $R(T)^W$-modules:

(i) $R(T) = \bigoplus_I R(T)_I$

(ii) $R(T)^{W_{\Delta \setminus I}} = \bigoplus_{J \subset I} R(T)_J$

Let

$$L := \bigoplus_{I \subset \Delta} \prod_{\alpha \in I}(1 - e^{-\alpha(u)})R(T) \otimes R(T)_I$$

For $I \subseteq \Delta$, the piece $\prod_{\alpha \in I}(1 - e^{-\alpha(u)})R(T) \otimes R(T)_I$ in the direct sum decomposition of $L$ is isomorphic to $R(T) \otimes R(T)_I$ as an $R(T) \otimes R(G)$-module, since $\prod_{\alpha \in I}(1 - e^{-\alpha(u)})$ is not a zero divisor in $R(T)$ (see Theorem 3.8 for details). Thus it follows from (i) above that $L$ is a free $R(T) \otimes R(G)$-module of rank $|W|$.

Further, by Lemma 3.2 it follows that $\prod_{\alpha \in I}(1 - e^{-\alpha(u)})R(T) \otimes R(T)_I \subseteq K_{G \times G}(X)$ for every $I \subseteq \Delta$. This is because, $\prod_{\alpha \in I}(1 - e^{-\alpha(u)})R(T) \otimes R(T)_I \subseteq R(T) \otimes R(T)$ clearly satisfies the relations which define $K_{G \times G}(X)$ in $R(T) \otimes R(T)$. In particular, when $I = \emptyset$, we get $R(T) \otimes R(G) \subseteq K_{G \times G}(X)$.

Let $K := K_{G \times G}(X)$. Thus we have the inclusion: $L \subseteq K$ as modules over $R(T) \otimes R(G)$.

Moreover, by Lemma 1.6 we know that $K$ is a free module over $R(G) \otimes R(G)$ of rank $|W|^2$. Further, note that $R(T) \otimes R(G)$ is free over $R(G) \otimes R(G)$ of rank $|W|$. Since $R(G) \otimes R(G)$ and $R(T) \otimes R(G)$ are regular, it follows that $K$ is a projective module over $R(T) \otimes R(G)$. Further, this implies that $K$ is free over $R(T) \otimes R(G)$ of rank $|W|$, by Theorem 1.1 of [12].

Thus, $L \hookrightarrow K \to K/L \to 0$ is a short exact sequence of $R(T) \otimes R(G)$ modules, and since $K$ and $L$ are free of rank $|W|$ it follows that $K/L$ is of projective dimension 1 as a module over $R(T) \otimes R(G)$.

We require to prove that $L = K$ as $R(T) \otimes R(G)$-modules. For this we first prove the following lemma.

**Lemma 3.4.** *Let $t_\alpha := \prod_{\beta \neq \alpha}(1 - e^{-\beta(u)}) \in R(T) \otimes R(G)$ for every $\alpha \in \Delta$. Then, $(K/L)_{t_\alpha} = 0$ for every $\alpha \in \Delta$.*

*Proof :* Let $M_\alpha := R(T) \otimes R(T)^{s_\alpha} \bigoplus (1 - e^{-\alpha(u)})R(T) \otimes e^{\omega_\alpha(v)}.R(T)^{s_\alpha}$.

Further note that $R(T) = R(T)^{s_\alpha} \bigoplus e^{\omega_\alpha} R(T)^{s_\alpha}$ for every $\alpha \in \Delta$ where $\omega_\alpha$ denotes the fundamental weight corresponding to $\alpha \in \Delta$. Hence, $R(T) \otimes R(T) = R(T) \otimes R(T)^{s_\alpha} \bigoplus R(T) \otimes e^{\omega_\alpha(v)} R(T)^{s_\alpha}$, which is in fact a direct sum decomposition of $R(T) \otimes R(T)$ as $R(T) \otimes R(T)^W$-module.

From Lemma 3.2 it follows that after localizing at $t_\alpha = \prod_{\beta \neq \alpha}(1-e^{-\beta(u)})$, the only condition defining $K_{G\times G}(X)$ in $R(T) \otimes R(T)$ is the one corresponding to $\alpha$. Using the above direct sum decomposition of $R(T) \otimes R(T)$ and the condition corresponding to $\alpha$, it follows that $K_{t_\alpha} \subseteq (M_\alpha)_{t_\alpha}$.

Moreover, from the equalities (i) and (ii) above, we get:
$R(T)^{s_\alpha} = \bigoplus_{\alpha \notin I} R(T)_I$
$e^{\omega_\alpha} \cdot R(T)^{s_\alpha} = \bigoplus_{\alpha \in I} R(T)_I$

Hence by the definition of $L$ it further follows that $L_{t_\alpha} = (M_\alpha)_{t_\alpha}$. Since $L_{t_\alpha} \subseteq K_{t_\alpha} \subseteq (M_\alpha)_{t_\alpha}$, we have: $(K/L)_{t_\alpha} = 0$ for every $\alpha \in \Delta$. □

Since the projective dimension of $(K/L) = 1$, by Auslander-Buchsbaum formula we know that *Supp* $(K/L)$ is of pure codimension 1 in *Spec* $(R(T) \otimes R(G))$. Hence *Supp* $(K/L)$ must contain a prime ideal $\mathfrak{p}$ of height 1 in $R(T) \otimes R(G)$. Since $R(T) \otimes R(G)$ is a U.F.D, $\mathfrak{p} = (a)$ for some $a \in R(T) \otimes R(G)$ and by Lemma 3.4 it follows that $\mathfrak{p}$ contains $1 - e^{-\alpha(u)}$ and $1 - e^{-\beta(u)}$ for $\alpha \neq \beta \in \Delta$.

This implies that $a$ divides $1 - e^{-\alpha(u)}$ and $1 - e^{-\beta(u)}$ for distinct $\alpha$ and $\beta$, which a contradiction since $1 - e^{-\alpha(u)}$ and $1 - e^{-\beta(u)}$ are relatively prime in the U.F.D, $R(T) \otimes R(G)$ (see p.182 of [3]).

This contradiction implies that $K/L = 0$ and hence $K = L$.

Now, for $I \subseteq \Delta$, the piece $\prod_{\alpha \in I}(1 - e^{-\alpha(u)})R(T) \otimes R(T)_I$ in the above direct sum decomposition is a free $R(T) \otimes R(G)$-module with basis $\{\prod_{\alpha \in I}(1 - e^{-\alpha(u)}) \otimes f_v \ : \ v \in C^I\}$ where $\{f_v\}$ is as in Notation 1.12. Thus the direct sum is a free $R(T) \otimes R(G)$-module of rank $|W|$ with basis $\{\prod_{\alpha \in I}(1 - e^{-\alpha(u)}) \otimes f_v \ : \ v \in C^I \textit{ and } I \subseteq \Delta\}$ □

For $I \subseteq \Delta$, let $A_I := \prod_{\alpha \in I}(1 - e^{-\alpha(u)})R(T) \otimes R(T)_I \subseteq R(T) \otimes R(T)$. The direct sum decomposition in Theorem 3.3 can therefore be expressed as:

$$K_{G\times G}(X) = \bigoplus_{I \subseteq \Delta} A_I.$$

**Corollary 3.5.** *The multiplicative structure of $K_{G\times G}(X)$ is determined by the above decomposition where the pieces $A_I$(resp. $A_{I'}$) corresponding to $I$ (resp. $I'$) multiply in $R(T) \otimes R(T)$ as follows:*

$$A_I \cdot A_{I'} \subseteq \prod_{\alpha \in I \cup I'} (1 - e^{-\alpha(u)}) R(T) \otimes R(T)^{W_{\Delta \setminus (I \cup I')}} \subseteq \bigoplus_{J \subseteq I \cup I'} A_J.$$

*In particular, any two basis elements $\prod_{\alpha \in I}(1 - e^{-\alpha(u)}) \otimes f_v$ and $\prod_{\alpha \in I'}(1 - e^{-\alpha(u)}) \otimes f_{v'}$, where $v$(resp. $v'$) belongs to $C^I$ (resp. $C^{I'}$) multiply in $R(T) \otimes$*

*$R(T)$ to give:*

$$(\prod_{\alpha\in I}(1-e^{-\alpha(u)})\otimes f_v)\cdot(\prod_{\alpha\in I'}(1-e^{-\alpha(u)})\otimes f_{v'}) = \prod_{\alpha\in I\cap I'}(1-e^{-\alpha(u)})\cdot \prod_{\alpha\in I\cup I'}(1-e^{-\alpha(u)})\otimes(f_v\cdot f_{v'})$$

*where the right hand side belongs to* $\prod_{\alpha\in I\cup I'}(1-e^{-\alpha(u)})R(T)\otimes R(T)^{W_{\Delta\setminus(I\cup I')}}$.

*Proof:* It follows by Lemma 3.2 that the direct sum decomposition $K_{G\times G}(X) = \bigoplus_{I\subseteq\Delta} A_I$ is a ring isomorphism where the multiplication on the right hand side is given as a subring of $R(T)\otimes R(T)$. We describe this multiplication below:

Let $B_I := \prod_{\alpha\in I}(1-e^{-\alpha(u)})R(T)\otimes R(T)^{W_{\Delta\setminus I}} \subseteq R(T)\otimes R(T)$. By Lemma 1.10 we have $A_I \subseteq B_I$ and further,

$$B_I = \bigoplus_{J\subseteq I}\prod_{\alpha\in I}(1-e^{-\alpha(u)})R(T)\otimes R(T)_J \subseteq \bigoplus_{J\subseteq I} A_J.$$

Moreover, we see that $B_I$ and $B_{I'}$ for $I, I' \subseteq \Delta$ multiply in $R(T)\otimes R(T)$ as follows:

$$B_I\cdot B_{I'} \subseteq B_{I\cup I'}.$$

Thus it follows that

$$A_I\cdot A_{I'} \subseteq B_{I\cup I'} \subseteq \bigoplus_{J\subseteq I\cup I'} A_J.$$

Hence the corollary. $\square$

Recall that since $G_{ad}$ is semisimple adjoint, $\Lambda_{ad} := X^*(T_{ad})$ has a basis consisting of the simple roots and further, since $G$ is semisimple and simply connected $\Lambda := X^*(T)$ has a basis consisting of the fundamental dominant weights. Thus $R(T_{ad}) = \mathbb{Z}[X^*(T_{ad})]$ is generated as a $\mathbb{Z}$-algebra by $\{e^{\alpha_i} : 1 \leq i \leq r\}$, and $R(T) = \mathbb{Z}[X^*(T)]$ is generated as a $\mathbb{Z}$-algebra by $\{e^{\omega_i} : 1 \leq i \leq r\}$.

Recall that on $X$ the isomorphism classes of line bundles correspond to $\lambda \in X^*(T)$. Further, the line bundle $\mathcal{L}_\lambda$ on $X$ associated to $\lambda$, admits a $G\times G$-linearisation so that $B^-\times B$ acts on the fibre $\mathcal{L}_\lambda\mid_z$ by the character $(\lambda, -\lambda)$, where $z$ denotes the base point of the unique closed orbit $G/B^-\times G/B$. Moreover, $Pic^{G\times G}(X)$ is freely generated by $\mathcal{L}_{\omega_i}$ corresponding to the fundamental dominant weights $\omega_i \in X^*(T)$ for $1\leq i\leq r$ (see §2.2 of [7]).

In particular, $\mathcal{L}_{\alpha_i}$ are $G\times G$-linearised line bundles such that $B^-\times B$ operates with the character $(\alpha_i, -\alpha_i)$ on $\mathcal{L}_{\alpha_i}\mid_z$ for $1\leq i\leq r$. Further, since the centre $\mathcal{Z}$ of $G$ acts trivially on $X$, and hence acts on the fibre by the character $(\alpha_i, -\alpha_i)$, $\mathcal{L}_{\alpha_i}$ is in fact $G_{ad}\times G_{ad}$-linearised. Moreover, $\mathcal{L}_{\alpha_i}$ also admits a $G_{ad}\times G_{ad}$-invariant section $s_i$ whose zero locus is the boundary divisor $X_i$ for $1\leq i\leq r$.

Moreover, since $\overline{T_{ad}}^+ \simeq \mathbb{A}^r$ where, each $I\subseteq\Delta$ corresponds to a $T_{ad}$-orbit $O_I = \{(x_1,\ldots,x_r)\in\mathbb{A}^r \mid x_i = 0 \text{ if and only if } \alpha_i\in I\}$. The base point $z$ of the closed $G_{ad}\times G_{ad}$-orbit in $X$ thus corresponds to the $T_{ad}\times T_{ad}$-fixed point $(0,\ldots,0)\in\mathbb{A}^r$. Further, let $(T_{ad})_I\subseteq T_{ad}$ denote the stabilizer at the orbit $O_I$.

Further, on $\overline{T_{ad}}^+$ the line bundle $\mathcal{L}_{\alpha_i}$ can be trivialised as a $T_{ad} \times T_{ad}$-equivariant line bundle $L_{\alpha_i} := \mathbb{A}^r \times \mathbb{C}$ where the $T_{ad} \times T_{ad}$-action is given by $(t_1, t_2) \cdot (u, c) = ((t_1, t_2) \cdot u \ , \ t_1^{\alpha_i} t_2^{-\alpha_i} \cdot c)$. In particular, we see that $diag(T_{ad})$ acts trivially on $L_{\alpha_i}$. The section $s_i$ further becomes the $i$th coordinate function whose zero locus is $\overline{O_{\{i\}}}$. Then $N_I := \bigoplus_{\alpha_i \in I} L_{\alpha_i}$ is the normal bundle of $\overline{O_I}$ in $\overline{T_{ad}}^+$, and hence

$$\lambda_{-1}(N_I^\vee) := \prod_{\alpha_i \in I} (1 - [L_{\alpha_i}^\vee]).$$

Note here that $L_{\alpha_i} = L_i$ in the notations of §2.1. With the above notations we have the following:

**Theorem 3.6.** *We have the following direct sum decomposition of $K_{G\times G}(X)$ as $R(T) \otimes R(G)$-module:*

$$K_{G\times G}(X) = \bigoplus_{I \subseteq \Delta} \lambda_{-1}(N_I^\vee) R(T) \otimes R(T)_I. \tag{3.1}$$

*Moreover, $K_{G\times G}(X)$ is free over $R(T) \otimes R(G)$ of rank $|W|$. Further, we can identify the component $R(T) \otimes 1 \subseteq R(T) \otimes R(T)^W$ in the above direct sum with the subring of $K_{G\times G}(X)$ generated by $\{[\mathcal{L}_\lambda] \ : \ \lambda \in \Lambda\}$, which is also the subring generated by $Pic^{G\times G}(X)$.*

*Proof:* The isomorphism (3.1) is an immediate consequence of Theorem 3.3. We can see this as follows: Since $diag(T)$ acts trivially on $L_{\alpha_i}$ we see that the isomorphism class of $L_{\alpha_i}$ in $K_{T\times T}(\overline{T}^+)$ corresponds to $e^{\alpha_i} \otimes 1$ in $R(T \times 1) \otimes R(diag(T)$. (Here we use the canonical identification, $R(T \times \{1\}) \otimes R(diag(T)) \simeq R(T) \otimes R(T)$.) Thus following the notations in Theorem 3.3, $[L_{\alpha_i}]$ corresponds to $e^{\alpha_i(u)}$. Therefore, by the definition of $N_I$ it follows that the term $\prod_{\alpha \in I}(1 - e^{-\alpha(u)})$, can be identified with $\lambda_{-1}(N_I^\vee)$ . Hence the claim.

By Prop. 2.5 and Remark 2.7 we have the inclusion $R(T) \times R(G) \subseteq K_{G\times G}(X)$. We claim that under the above inclusion the image of $R(T) \otimes 1$ in $K_{G\times G}(X)$ is the subring generated by $\{[\mathcal{L}_\lambda] \ : \ \lambda \in \Lambda\}$ which is the subring generated by the isomorphism classes of $G \times G$-linearised line bundles on $X$. This can be seen as follows:

By Cor.2.3 and Remark 2.7 we have the canonical inclusion $K_{G\times G}(X) \hookrightarrow R(T) \otimes R(T)$ obtained by restriction to the base point $z$ of the unique closed orbit $G/B^- \times G/B$. By definition, $\mathcal{L}_\lambda$ maps to $e^\lambda \otimes e^{-\lambda}$ under the above inclusion. Further, by the canonical identification $R(T \times \{1\}) \otimes R(diag(T)) \simeq R(T) \otimes R(T)$ coming from the exact sequence $1 \to diag(T) \to T \times T \to T \times \{1\} \to 1$, we see that the image of $[\mathcal{L}_\lambda]$ under the restriction map is $e^\lambda \otimes 1$ for $\lambda \in \Lambda$. Since $e^\lambda \otimes 1$ generate $R(T) \otimes 1$, it follows that the image of $R(T) \otimes 1$ under the above restriction in $R(T) \otimes R(T)$ is same as the image of the subring generated by $[\mathcal{L}_\lambda]$ for $\lambda \in \Lambda$. Further, since $[\mathcal{L}_\lambda]$ for $\lambda \in \Lambda$ generate $Pic^{G\times G}(X)$, we have the theorem. $\square$

**Remark 3.7.** Note that we can also identify $R(T_{ad})\otimes 1 \subseteq R(T_{ad})\otimes R(T_{ad})^W \subseteq K_{G_{ad}\times G_{ad}}(X)$ with the subring of $K_{G_{ad}\times G_{ad}}(X)$ generated by $\{[\mathcal{L}_{\alpha_i}] : 1 \leq i \leq r\}$. Thus $K_{G_{ad}\times G_{ad}}(X)$ is a module over the subring generated by the isomorphism classes of the $G_{ad}\times G_{ad}$-linearised line bundles on $X$ corresponding to the boundary divisors $\{X_i : 1 \leq i \leq r\}$.

In $R(T)$ let

$$f_v \cdot f_{v'} = \sum_{J\subseteq (I\cup I')} \sum_{w\in C^J} a^w_{v,v'} \cdot f_w \tag{3.3}$$

for certain elements $a^w_{v,v'} \in R(G) = R(T)^W \ \ \forall\ v \in C^I$, $v' \in C^{I'}$ and $w \in C^J$, $J \subseteq (I \cup I')$ (see Notation 1.12).

**Theorem 3.8.** *We have the following isomorphism as $R(T) \otimes R(T)^W$-submodules of $R(T) \otimes R(T)$.*

$$\bigoplus_{I\subseteq\Delta} R(T)\otimes R(T)_I \simeq \bigoplus_{I\subseteq\Delta} \lambda_{-1}(N_I^\vee)R(T)\otimes R(T)_I = K_{G\times G}(X).$$

*More explicitly, the above isomorphism maps an arbitrary element $a \otimes b \in R(T)\otimes R(T)_I$ to the element $(\lambda_{-1}(N_I^\vee)\cdot a)\otimes b$. In particular, the basis element $1 \otimes f_v \in R(T)\otimes R(T)_I$ maps to $\lambda_{-1}(N_I^\vee)\otimes f_v \in \lambda_{-1}(N_I^\vee)\cdot R(T)\otimes R(T)_I$, for $v \in C^I$ for every $I \subseteq \Delta$.*

*We now define a multiplication on $\bigoplus_{I\subseteq\Delta} R(T)\otimes R(T)_I$ where any two basis elements $1\otimes f_v$ and $1\otimes f_{v'}$ for $v \in C^I$, $v' \in C^{I'}$ $(I, I' \subseteq \Delta)$ multiply as follows:*

$$(1\otimes f_v)\cdot(1\otimes f_{v'}) := \sum_{J\subseteq(I\cup I')}\sum_{w\in C^J} (\lambda_{-1}(N^\vee_{I\cap I'})\cdot \lambda_{-1}(N^\vee_{(I\cup I')\setminus J})\otimes a^w_{v,v'})\cdot(1\otimes f_w).$$

*Then the above isomorphism further preserves the multiplicative structure where on the right hand side the multiplication is as defined in Cor. 3.5.*

*Proof:* Note that we have a canonical isomorphism of $R(T)$ with $K_T(\overline{T}^+)$ which maps $e^{\alpha_i}$ to $[L_{\alpha_i}]$ for $1 \leq i \leq r$. Thus by the notations in Theorem 3.3 we have the following identification:

$$\prod_{\alpha\in I}(1-e^{-\alpha(u)})R(T)\otimes R(T)_I = \lambda_{-1}(N_I^\vee)R(T)\otimes R(T)_I,$$

where the basis element $\prod_{\alpha\in I}(1-e^{-\alpha(u)})\otimes f_v$ corresponds to $\lambda_{-1}(N_I^\vee)\otimes f_v$ for $v \in C^I$ for every $I \subseteq \Delta$. Further, in the direct sum decomposition:

$$K_{G\times G}(X) = \bigoplus_{I\subseteq\Delta} \lambda_{-1}(N_I^\vee)R(T)\otimes R(T)_I,$$

the multiplication of two basis elements $\lambda_{-1}(N_I^\vee)\otimes f_v$ and $\lambda_{-1}(N_{I'}^\vee)\otimes f_{v'}$, where $v$(resp. $v'$) belongs to $C^I$ (resp. $C^{I'}$) given in Cor.3.5 can be expressed as follows:

$$(\lambda_{-1}(N_I^\vee)\otimes f_v)\cdot(\lambda_{-1}(N_{I'}^\vee)\otimes f_{v'})=\lambda_{-1}(N_{I\cap I'}^\vee)\cdot\lambda_{-1}(N_{I\cup I'}^\vee)\otimes f_v\cdot f_{v'} \quad (3.4)$$

Note that the right hand side of the above equality (3.4) belongs to

$$\lambda_{-1}(N_{I\cup I'}^\vee)\cdot R(T)\otimes R(T)^{W_{\Delta\setminus(I\cup I')}}\subseteq\bigoplus_{J\subseteq(I\cup I')}\lambda_{-1}(N_J^\vee)\cdot R(T)\otimes R(T)_J,$$

and further using (3.3) the right hand side of (3.4) can be rewritten as follows:

$$\sum_{J\subseteq(I\cup I')}\sum_{w\in C^J}(\lambda_{-1}(N_{I\cap I'}^\vee)\lambda_{-1}(N_{(I\cup I')\setminus J}^\vee)\otimes a_{v,v'}^w)\cdot(\lambda_{-1}(N_J^\vee)\otimes f_w). \quad (3.5)$$

Recall that $\lambda_{-1}(N_I^\vee)$ is not a zero divisor in $K_T(\overline{T_{ad}}^+)$ (see Lemma 4.2 of [23] and proof of Cor. 2.12). Thus we see that each piece $R(T)\otimes R(T)_I$ is isomorphic to $\lambda_{-1}(N_I^\vee)\cdot R(T)\otimes R(T)_I$ for every $I\subseteq\Delta$, as $R(T)\otimes R(G)$-submodules of $R(T)\otimes R(T)$, where the isomorphism maps an element $a\otimes b\in R(T)\otimes R(T)_I$ to the element $(\lambda_{-1}(N_I^\vee)\cdot a)\otimes b\in\lambda_{-1}(N_I^\vee)\cdot R(T)\otimes R(T)_I$.

Further, this additively extends to an isomorphism of $R(T)\otimes R(T)^W$-submodules of $R(T)\otimes R(T)$:

$$\bigoplus_{I\subseteq\Delta}R(T)\otimes R(T)_I\simeq\bigoplus_{I\subseteq\Delta}\lambda_{-1}(N_I^\vee)R(T)\otimes R(T)_I.$$

Now by the definition of multiplication in $\bigoplus_{I\subseteq\Delta}R(T)\otimes R(T)_I$ we have:

$$(1\otimes f_v)\cdot(1\otimes f_{v'}):=\sum_{J\subseteq(I\cup I')}\sum_{w\in C^J}(\lambda_{-1}(N_{I\cap I'}^\vee)\cdot\lambda_{-1}(N_{(I\cup I')\setminus J}^\vee)\otimes a_{v,v'}^w)\cdot(1\otimes f_w).$$

Thus it follows that under the above isomorphism $(1\otimes f_v)\cdot(1\otimes f_{v'})$ maps to

$$\sum_{J\subseteq(I\cup I')}\sum_{w\in C^J}(\lambda_{-1}(N_{I\cap I'}^\vee)\lambda_{-1}(N_{(I\cup I')\setminus J}^\vee)\otimes a_{v,v'}^w)\cdot(\lambda_{-1}(N_J^\vee)\otimes f_w)$$

which by (3.5) is equal to $(\lambda_{-1}(N_I^\vee)\otimes f_v)\cdot(\lambda_{-1}(N_{I'}^\vee)\otimes f_{v'})$ in $\bigoplus_{I\subseteq\Delta}\lambda_{-1}(N_I^\vee)R(T)\otimes R(T)_I$. Hence the theorem. $\square$

**Remark 3.9.** Note that, since $\lambda_{-1}(N_\emptyset^\vee)=1$, under the isomorphism

$$\bigoplus_{I\subseteq\Delta}R(T)\otimes R(T)_I\simeq\bigoplus_{I\subseteq\Delta}\lambda_{-1}(N_I^\vee)R(T)\otimes R(T)_I$$

defined in Theorem 3.8 the piece $R(T)\otimes R(G)=R(T)\otimes R(T)_\emptyset$ maps isomorphically onto itself. In particular, the subring generated by $Pic^{G\times G}(X)$ in $K_{G\times G}(X)$, which is canonically identified with $R(T)\otimes\{1\}$ on the right hand side of the above isomorphism, maps isomorphically onto $R(T)\otimes\{1\}\subseteq R(T)\otimes R(T)_\emptyset$ in the direct sum $\bigoplus_{I\subseteq\Delta}R(T)\otimes R(T)_I$.

**Remark 3.10.** Note that Theorem 3.8 gives an explicit description of the *multiplicative structure constants* in terms of the basis $\{1\otimes f_v : \ v \in C^I, I \subseteq \Delta\}$ for $K_{G\times G}(X)$. This further enables us to *directly apply* this description for the multiplicative structure of ordinary $K$-ring of $X$ in the following section (see Theorem 3.12 and Remark 3.13).

## 3.1 Ordinary K-ring of the wonderful compactification

Let $X$ denote the wonderful compactification of $G_{ad}$. We follow the notations of §3 (see Notation 3.1).

Further, since $K_G(G/B) \simeq R(T)$ and $K_G(pt) = R(G)$, the characteristic map $R(T) \to K(G/B)$ induces an isomorphism $R(T)/\mathcal{J} \simeq K(G/B)$ where $\mathcal{J}$ denotes the ideal generated by $\{f - \epsilon(f) | f \in R(T)^W\}$, where $\epsilon : R(T) \to \mathbb{Z}$ is the augmentation map given by $\epsilon(e^{\lambda}) = 1$ for $\lambda \in \Lambda$.

**Lemma 3.11.** *Let $\varphi : R(T) \to K(G/B)$ denote the characteristic map, and $\overline{f}_v = \varphi(f_v)$ for every $v \in W$. Then $\overline{f}_v$ for $v \in W$ form a basis of $R(T)/\mathcal{J} = K(G/B)$ over $\mathbb{Z}$.*

*Proof:* Recall from Notation 1.12 that $\{f_v\}_{v\in W}$ form a basis for $R(T)$ as $R(T)^W$-module. Since the characteristic map $\varphi : R(T) \to K(G/B)$ is surjective, $\{\overline{f}_v : v \in W\}$ generate $K(G/B)$ as $\mathbb{Z}$-module. Further, we claim that $\{\overline{f}_v\}_{v\in W}$ are linearly independent over $\mathbb{Z}$. This can be seen as follows:

Let $\sum_{v\in W} b_v \cdot \overline{f}_v = 0$ for some $b_v \in \mathbb{Z}$. Now, since $\varphi\mid_{R(G)}: R(G) \to \mathbb{Z}$ is surjective, $b_v = \varphi(a_v)$ for some $a_v \in R(G)\ \forall\ v \in W$. Thus we see that $\sum_{v\in W} a_v \cdot f_v = c \in \mathcal{J}$. Further, recall that $\mathcal{J} \subseteq R(G)$, and $\{f_v : v \in W\}$ are linear independent over $R(G)$, where $f_1 = 1 \in R(G)$. Thus it follows that $a_1 = c$ and $a_v = 0$ for all $v \neq 1$. This further implies that $b_v = 0$ for every $v \in W$. Hence the claim. □

Further, let

$$K(G/B)_I := \bigoplus_{v\in C^I} \mathbb{Z}[\overline{f}_v],$$

Then we have:

$$K(G/B) = \bigoplus_{I\subseteq\Delta} K(G/B)_I.$$

In this section we denote the image in $K(G/B)$ of $e^{\alpha} \in R(T)$ under the characteristic map by $[L_{\alpha}]$. Further, we shall denote by the same symbol $\lambda_{-1}(N_I^{\vee}) \in K(G/B)$, the image of $\lambda_{-1}(N_I^{\vee}) = \prod_{\alpha\in I}(1 - e^{-\alpha}) \in R(T)$ for every $I \subseteq \Delta$. (However, note that since $\lambda_{-1}(N_I^{\vee}) \in \mathcal{J}$, its image in $K(G/B)$ is always nilpotent.) Furthermore, we let $\overline{a}^w_{v,v'} \in \mathbb{Z}$ denote the image under $\varphi\mid_{R(G)}$ of the element $a^w_{v,v'} \in R(G) = R(T)^W$ defined in (3.3).

**Theorem 3.12.** *We have a canonical $K(G/B)$-module structure on $K(X)$, induced from the $R(T)\otimes 1$-module structure on $K_{G\times G}(X)$ given in Theorem 3.6. Moreover, $K(X)$ is a free module of rank $|W|$ over $K(G/B)$, $K(G/B)$ being identified with the subring of $K(X)$ generated by $Pic(X)$.*

*More explicitly, let*

$$\gamma_v := 1 \otimes \overline{f}_v \in K(G/B) \otimes K(G/B)_I$$

*for $v \in C^I$ for every $I \subseteq \Delta$. Then we have:*

$$K(X) \simeq \bigoplus_{v\in W} K(G/B)\cdot \gamma_v.$$

*Further, the above isomorphism is a ring isomorphism, where the multiplication of any two basis elements $\gamma_v$ and $\gamma_{v'}$ is defined as follows:*

$$\gamma_v \cdot \gamma_{v'} := \sum_{J\subseteq (I\cup I')} \sum_{w\in C^J} (\lambda_{-1}(N^{\vee}_{I\cap I'})\cdot \lambda_{-1}(N^{\vee}_{(I\cup I')\setminus J})\cdot \overline{a}^{w}_{v,v'})\cdot \gamma_w.$$

*Proof:* By Theorem 3.8 we have the following direct sum decomposition of $K_{G\times G}(X)$ as an $R(T)\otimes R(G)$-module:

$$K_{G\times G}(X) \simeq \bigoplus_{I\subseteq\Delta} R(T)\otimes R(T)_I.$$

Now, using isomorphism $(c)$ of §1.2 for $G\times G$ we get:

$$K(X) \simeq \bigoplus_{I\subseteq\Delta} K(G/B)\otimes K(G/B)_I.$$

Further, under the canonical restriction homomorphism $K_{G\times G}(X)\to K(X)$, the image of the subring generated by $Pic^{G\times G}(X)$ in $K_{G\times G}(X)$, maps surjectively onto the subring generated by $Pic(X)$ in $K(X)$. Hence by Remark 3.9, it follows that under the above isomorphism, the subring generated by $Pic(X)$ in $K(X)$ maps isomorphically onto the piece $K(G/B)\otimes 1 \subseteq K(G/B)\otimes K(G/B)_{\emptyset}$.

Let

$$\gamma_v := 1 \otimes \overline{f}_v \in K(G/B) \otimes K(G/B)_I$$

for $v \in C^I$ for every $I\subseteq\Delta$. Then $\gamma_v$ is the image of the element $1\otimes f_v \in R(T)\otimes R(T)_I$ in $K(G/B)\otimes K(G/B)_I$ under the characteristic map.

Then identifying $K(G/B)\otimes 1 \simeq K(G/B)$, we have:

$$K(X) \simeq \bigoplus_{v\in W} K(G/B)\cdot\gamma_v. \tag{3.6}$$

Thus we see that $K(X)$ is a free module of rank $|W|$ over $K(G/B)$ with basis $\gamma_v$ for $v \in W$, $K(G/B)$ being identified with the subring of $K(X)$ generated by $Pic(X)$.

Recall from Theorem 3.8 that the multiplication of two basis elements $(1 \otimes f_v)$ and $(1 \otimes f_{v'})$ of $K_{G\times G}(X) \simeq \bigoplus_{I\subseteq\Delta} R(T) \otimes R(T)_I$ is defined as:

$$(1 \otimes f_v) \cdot (1 \otimes f_{v'}) := \sum_{J\subseteq(I\cup I')} \sum_{w\in C^J} (\lambda_{-1}(N^\vee_{I\cap I'}) \cdot \lambda_{-1}(N^\vee_{(I\cup I')\setminus J}) \otimes a^w_{v,v'}) \cdot (1 \otimes f_w).$$

Thus their images, $\gamma_v$ and $\gamma_{v'}$ in $\bigoplus_{I\subseteq\Delta} K(G/B) \otimes K(G/B)_I$, multiply as follows:

$$\gamma_v \cdot \gamma_{v'} := \sum_{J\subseteq(I\cup I')} \sum_{w\in C^J} (\lambda_{-1}(N^\vee_{I\cap I'}) \cdot \lambda_{-1}(N^\vee_{(I\cup I')\setminus J}) \cdot \overline{a}^w_{v,v'}) \cdot \gamma_w. \tag{3.7}$$

Thus we conclude that the above isomorphism (3.6) is further a ring isomorphism, where the multiplication of any two basis elements $\gamma_v$ and $\gamma_{v'}$ for $v \in C^I$, $v' \in C^{I'}$ and $I, I' \subseteq \Delta$ is defined as in (3.7).

Hence the theorem. □

**Remark 3.13.** Note that in the direct sum decomposition of $K_{G\times G}(X)$ given in Theorem 2.12, each piece of the direct sum is canonically isomorphic to $R(T) \otimes R(G)$-submodules of $R(T) \otimes R(T)$ (see Theorem 3.8). This enables us in the equivariant setup to describe the multiplication of the direct sum pieces, and hence the basis elements inside the subring $R(T) \otimes R(T)$ (see Cor. 3.5 and Theorem 3.8). However, this cannot be done in ordinary $K$-theory since the image of $\lambda_{-1}(N_I^\vee)$ under the characteristic homomorphism becomes nilpotent in the ordinary $K$-ring. Hence the multiplication of the basis elements in ordinary $K$-ring needs to be defined suitably by pushing down the multiplicative structure from the equivariant $K$-ring.

**Concluding Remarks:**

1. *Extending the results to arbitrary fields and higher $K$-theory* :

   We believe that the results in this paper should hold over any algebraically closed field of arbitrary characteristic. It is also likely that many of the results hold in the setting of higher $K$-theory.

2. *Geometric interpretation of the basis $\{f_v\}_{v\in W}$* :

   By Prop. 1.9 it follows that when $v \in W^{\Delta\setminus I}$,

$$\overline{f_v} = \sum_{x\in W'_{\Delta\setminus I}(v)} \varphi(f^{\emptyset}_{vx})$$

where $\overline{f_v}$ and $\varphi$ are as in Lemma 3.11. Further, by (1.5) we see that $\varphi(f_{vx}^{\emptyset})$ is the class of the line bundle in $K(G/B)$ corresponding to the weight

$$v^{-1}\big(\sum_{v^{-1}\alpha_i<0}\omega_i\big).$$

We are now trying to obtain a more comprehensive geometric interpretation of the basis elements $f_v$ and $\overline{f_v}$ (and of the Steinberg basis). Such an interpretation may be well known to experts but we were unable to find it in the literature.

3. In a private communication we were informed by Prof. De Concini that E. Strickland has recently determined the structure of the cohomology ring of the wonderful compactification.

# References


[1] J.F. Adams, *Lectures on Lie Groups*, Benjamin, 1967.

[2] E. Bifet, C. De Concini and C. Procesi, Cohomology of regular embeddings, *Adv. Math.* **82** (1990), 1-34.

[3] N. Bourbaki, *Groupes et algebres de Lie*, Hermann, Paris, 1968.

[4] M. Brion, The behaviour at infinity of the Bruhat decomposition, *Comment. Math. Helv.* **73** (1998), 137-174.

[5] M. Brion, Equivariant Chow Groups for Torus Actions, *Journal of Transformation Groups* **2** (1997), 225-267.

[6] M. Brion, Positivity in the Grothendieck group of complex flag varieties, *J. Algebra* **258** (2002), 137-159.

[7] M. Brion, Groupe de Picard et nombres caracteristiques des varietes spheriques, *Duke Math. J.***58**, No. 2, 397-424, 1989.

[8] N.Chriss and V.Ginzburg, *Representation theory and complex geometry*, Birkhauser (1997).

[9] C.W.Curtis and I.Reiner, *Methods of Representation Theory Vol. 1. With applications to finite groups and orders.*, John Wiley & Sons Inc., New York (1981).

[10] C. De Concini and C. Procesi, Complete symmetric varieties, *Invariant Theory* (Proceedings, Montecatini 1982), *Lecture Note in Math.* **996**, 1-44, Springer-Verlag, New York 1983.

[11] C. De Concini and C. Procesi, Cohomology of compactification of algebraic groups, *Duke Mathematical Journal*, **53**, No. 3 (1986), 585-594.

[12] J. Gubeladze, K-theory of affine toric varieties, *Homology, Homotopy and Applications* **1** (1999), 135-145 .

[13] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press (1990).

[14] B. Iversen, The Geometry of Algebraic Groups, *Adv. Math.* **20** (1976), 57-85.

[15] A. Joseph, On the Demazure character formula, *Ann. Sci. Ecole Norm. Super.* **18** (1985), 389-419.

[16] P. Littelmann and C. Procesi, Equivariant Cohomology of Wonderful Compactifications, *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory*, Birkhauser, Basel 1990.

[17] R.Marlin, Anneaux de Grothendieck et varietes de drapeaux, *Bull. Soc. math. France*, **104**, (1976), 337-348.

[18] John McLeod, The Kunneth Formula in equivariant K-theory, *Lecture Notes in Math.*, **741** 316-333, Springer Verlag, Berlin, 1979.

[19] A.S. Merkurjev, Comparison of eqivariant and ordinary K-theory of algebraic varieties, *Algebra i Anliz* **9** (1997), 175-214. *Translation in St. Petersburg Math. J.* **9** (1998), 815-850.

[20] H.V.Pittie, Homogeneous vector bundles on homogeneous spaces, *Topology* **11** (1972), 199-203.

[21] P.Sankaran and V.Uma, Cohomology of toric bundles, *Comment. Math. Helv.*, **78** (2003), 540-554. Errata, **79** (2004), 840-841.

[22] R. Steinberg, On a theorem of Pittie, *Topology* **14** (1975), 173-177.

[23] G.Vezzosi and A.Vestoli, Higher algebraic K-theory for actions of diagonalizable groups, *Invent. Math.* **153** (2003), 1-44.

Address:
Institut Fourier
100, rue des Maths, BP74
38402 St Martin d'Heres
Cedex, FRANCE
E-mail:`uma@mozart.ujf-grenoble.fr`

Current Address:
Department of Mathematics
Indian Institute of Technology, Madras
Chennai-600036, INDIA.
E-mail:`vuma@iitm.ac.in`
`uma@cmi.ac.in`